Characterizations of invertible matrices

Math 4A – Scharlemann

4 February 2015

CELL PHONES OFF

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A given matrix may or may not be invertible, but if it is invertible, then there are an amazing number of equivalent facts true about the matrix, as we shall see.

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4. *The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.*
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6. *Linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one (injective)*
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Suppose when put in reduced echelon form, the matrix becomes I_n.
We have seen that the same row operations applied to I_n will give A^{-1}, so A is invertible.

We have just shown that these are equivalent:

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12. A^T is invertible

iClicker question

Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

invertible?

A Yes, the theorem says it is.

B It's easy to see the theorem says it is not.

C You can't easily say without doing a long calculation.

Discussion

You may have noticed that

$$-2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So 4) is false. Since 4) is equivalent to A being invertible, A is not invertible.

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Theorem

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$$(A^T)^{-1} = (A^{-1})^T$$

Proof: Suppose A is invertible and C is its inverse. Then $AC = I_n = CA$.

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Now substitute $I_n^T = I_n$ in the middle to get

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This shows that C^T is an inverse for A^T.

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Easy to reverse the argument: If $AD = I_n$ then A invertible.

We now have that these are equivalent:

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4. The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
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invertible?

A Yes, the theorem says it is.

B It's easy to see the theorem says it is not.

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Discussion

Observe that

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}^T = A^T.$$

Since we know that A is not invertible, it follows from 10) that $A^T = B$ is not invertible.

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Since the first is black, we can color the second black.

By the same argument, these three statements are equivalent (but none is yet black)

- For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
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This leaves us at:

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Suppose there is a D so that $AD = I_n$. Then for any $\vec{b} \in \mathbb{R}^n$, let $\vec{x} = D\vec{b}$ and discover that $A\vec{x} = A(D\vec{b}) = (AD)\vec{b} = I_n\vec{b} = \vec{b}$.

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To complete an equivalence, we need to show:

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Now set

$$D = \begin{bmatrix} \vec{d}_1 & \vec{d}_2 & \dots & \vec{d}_n \end{bmatrix}$$

Then

$$AD = A\begin{bmatrix} \vec{d}_1 & \vec{d}_2 & \dots & \vec{d}_n \end{bmatrix} = \begin{bmatrix} A\vec{d}_1 & A\vec{d}_2 & \dots & A\vec{d}_n \end{bmatrix} =$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{bmatrix} = I_n$$

iClicker question

Suppose I tell you the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the solution $\vec{x} = \vec{0}$. What can you say about the equation

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- A There is at least one solution $\vec{x}$.
- B There is at most one solution $\vec{x}$.
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For each $\vec{b}$ there is a solution to the equation $A\vec{x} = \vec{b}$.
So option A), there is at least one solution to the equation
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Once we see that the last is also equivalent, option B) is also true:
There is at most one solution $\vec{x}$.

The final item is brought into the equivalence via:

Theorem

These are equivalent:

- *The only solution to the equation* $A\vec{x} = \vec{0}$ *is* $\vec{x} = \vec{0}$.
- *Linear transformation* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is one-to-one (injective)*

Proof: Suppose A is injective. Then whenever $A\vec{x} = \vec{0}$ we have $A\vec{x} = \vec{0} = A\vec{0}$). But injectivity then implies $\vec{x} = \vec{0}$.

The final item is brought into the equivalence via:

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So $\vec{u} - \vec{w} = \vec{0}$, which means $\vec{u} = \vec{w}$ and $\vec{u}, \vec{w}$ are not different.