MULTIPLE GENUS 2 HEEGAARD SPLITTINGS: A MISSED CASE

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Abstract. A gap in [RS] is filled: new examples are found of closed orientable 3-manifolds with possibly multiple genus 2 Heegaard splittings. Properties common to all the examples in [RS] are not universally shared by the new examples: some of the new examples have Hempel distance 3, and it is not clear that a single stabilization always makes the multiple splittings isotopic.

1. Introduction

In 1998, Rubinstein and the second author [RS] studied the question of when there could be more than one distinct genus 2 Heegaard splitting of the same closed orientable 3-manifold. The goal of the project was modest: to provide a complete list of ways in which such multiple splittings could be constructed, but with no claim that each example on the list did in fact have multiple non-isotopic splittings (there could be isotopies from one splitting to another that are not apparent). Nor was there a claim that the list had no redundancies; a 3-manifold and its multiple splittings might appear more than once on the list. Such a list would still be useful, for if every example on the list could be shown to have a certain property, then that property would be true for any closed orientable 3-manifold M that has multiple genus 2 splittings. Two examples were given in [RS]:

- If M is atoroidal then the hyperelliptic involutions determined by the two genus 2 Heegaard splittings commute.
- Any two genus 2 Heegaard splittings of M become isotopic after a single stabilization.

Despite this modest goal, the argument in [RS] contains a gap. In 2008, the first author discovered a class of examples that do not appear on the list and which, moreover, have mathematical properties that distinguish them in important ways from the examples that do

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appear in [RS]. It is true that, even for the new examples, the hyperelliptic involutions commute. But we know of no argument showing that the new examples all share the second property above; that is, we cannot show that the newly discovered multiple splittings necessarily become isotopic after a single stabilization (though they do after two stabilizations).

A third property, shared by all examples in [RS] but not by some of the new examples, is not listed above because the notion of Hempel distance of Heegaard splittings (see [He]) did not exist at the time [RS] was written. But a retrospective look (see Section 7 below) will verify that all the splittings described in [RS] have Hempel distance no greater than 2, whereas results of the first author [Be2] illustrate that at least some of the new examples have Hempel distance 3. (This also verifies that the gap in the argument in [RS] actually led to missed examples.)

The present paper serves as an erratum to [RS][1] and describes the new examples. Here is an outline: In Section 2 we describe a general method for constructing closed orientable 3-manifolds that appear to have multiple genus 2 Heegaard splittings; these examples (called *Dehn-derived*) are based around Dehn surgery on a pair of strategically placed curves. It follows from the construction that always the hyperelliptic involutions of the alternate splittings coincide.

It is not immediately obvious that curves supporting Dehn-derived examples can be found, but in Sections 3 and 4 we give three specific classes of examples. The classes are denoted M_H (Section 3), $M_{\times I}$ and M_{hybrid} (Section 4). (M_H can be viewed as a third variation of [RS, Example 4.2].) For the examples M_H and M_{hybrid} a single stabilization suffices to make the alternate splittings equivalent, but this property is at least not apparent in most cases of $M_{\times I}$.

Section 5 describes how the proof of [RS, Lemma 9.5, Case 2] went astray and how it needs to be altered to fix the gap. The upshot is that Dehn-derived examples, as described in Section 2, do fill the gap in the original proof. In Section 6 it is further shown, using new results in [Be2], that *any* Dehn-derived example is in fact of type M_H, $M_{\times I}$ or M_{hybrid}. Finally, in Section 7 we verify that all of the old examples that are listed in [RS] are of Hempel distance 2, whereas at least some Dehn-derived examples are of distance 3. (It is easy to see that all Dehn-derived examples are of distance no more than 3.)

[1]The error is on p. 533: The last sentence of the first paragraph of Case 2 should have read, "The *same curves* cannot then be twisted in X since M is hyperbolike." This leaves open an additional possibility for P_X, P_Y, which appears as Subcase B in Section 5 below.

2. Dehn derived multiple splittings

A *primitive k-tuple* of curves in the boundary of a genus g handlebody H is a collection $\lambda_1, ...\lambda_k \subset \partial H$ of $k \leq g$ disjoint simple closed curves so that, for some properly embedded collection $D_1, ..., D_k$ of disks in H, $|\lambda_i \cap D_j| = \delta_{ij}, 1 \leq i, j \leq k$. It is easy to see that the closed complement in H of such a collection of meridian disks is a genus $g-k$ handlebody. In particular, if $k = g$ then $\lambda_1, ...\lambda_g$ is called a *complete set of primitive curves* and the corresponding collection of disks $D_1, ..., D_g$ is called a *complete set of meridian disks*. The closed complement of a complete set of meridian disks in H is a 3-ball.

Suppose $\Lambda = \lambda_1, ...\lambda_k \subset \partial H$ is a primitive k-tuple of curves in H and let $\alpha_1, ..., \alpha_k$ be the properly embedded collection of curves in H obtained by pushing Λ slightly into the interior of H. We can view H as the boundary connect sum of a genus $g-k$ handlebody H' and k solid tori $W_1, ..., W_k$, with λ_i a longitude of W_i and so α_i a core curve of W_i. Then Dehn surgery on $\alpha_i \subset W_i$ still gives a solid torus. Hence any Dehn surgery on the family of curves $\alpha_1, ..., \alpha_k$ leaves H still a handlebody.

Definition 2.1. *Suppose $M_0 = H_a \cup_S H_b$ is a Heegaard splitting of a closed 3-manifold M_0. A simple closed curve $\lambda \subset S$ is* doubly primitive *if λ is a primitive curve in both handlebodies H_a and H_b.*

Suppose M_0 is a closed orientable 3-manifold and that $M_0 = H_a \cup_S H_b$ is a genus 2 Heegaard splitting of M_0. Suppose further that $\lambda_1, \lambda_2 \subset S$ are two disjoint doubly primitive curves in S.

Proposition 2.2. *Suppose M is a manifold obtained by some specified Dehn surgeries on λ_1 and λ_2. For $i = 1, 2$, let A_i (resp. B_i) be the manifold obtained from the handlebody H_a (resp. H_b) by pushing the curve λ_i into $\mathrm{int}(H_a)$ (resp $\mathrm{int}(H_b)$) and performing the specified Dehn surgery on the curve.*

Then $A_1 \cup_S B_2$ and $A_2 \cup_S B_1$ are two (possibly different) genus 2 Heegaard splittings of M.

Proof. A_i (resp B_i) is obtained from H_a (resp H_b) by Dehn surgery on a pushed in copy α_i of a single primitive curve in S. It was just observed that this makes each A_i (resp. B_i) a handlebody. □

Definition 2.3. *Two genus 2 Heegaard splittings $X \cup_Q Y$ and $A \cup_P B$ of a closed 3-manifold M are called* Dehn derived *(from the splitting $M_0 = H_a \cup_S H_b$ via $\lambda_1 \cup \lambda_2 \subset S$) if the two splittings are created as in Proposition 2.2.*

Corollary 2.4. *Suppose $M = A \cup_P B = X \cup_Q Y$ are a Dehn-derived pair of Heegaard splittings. Then the two hyperelliptic involutions of M, one determined by the Heegaard splitting $A \cup_P B$ and the other by the Heegaard splitting $X \cup_Q Y$, coincide.*

Proof. Let $M_0 = H_a \cup_S H_b$ be the Heegaard split 3-manifold from which the two splittings of M are Dehn derived, via $\lambda_1 \cup \lambda_2 \subset S$. The hyperelliptic involution preserves the isotopy class (though perhaps reversing the orientation) of any simple closed curve in S. We may then position λ_i so that the curves are preserved (reversing orientation) by the hyperelliptic involution on $M_0 = H_a \cup_S H_b$. Then the hyperelliptic involution on M_0 naturally induces a single hyperelliptic involution on M. □

3. A simple set of examples

It is not immediately obvious how to create examples of a Dehn-derived pair of splittings or, very naively, whether examples even exist. In this section we present and briefly discuss an important concrete class of examples.

Consider a genus 2 handlebody H, constructed from two 0-handles by connecting them with three 1-handles. With this structure H has a natural $\mathbb{Z}_3$ symmetry, shown as $\frac{2\pi}{3}$ rotation about the green axis in Figure 1. Let $\lambda_1 \subset \partial H$ be the red curve shown in the figure and λ_2, λ_3 be the other two simple closed curves to which λ_1 is carried by the $\mathbb{Z}_3$ symmetry. Then each λ_i is a primitive curve on ∂H and, indeed, any two of the curves, say λ_1, λ_2 constitute a complete set of primitive curves (that is, a primitive pair). In this case the corresponding pair of meridian disks are the meridian disks of the two 1-handles through which λ_3 passes.

Let $\overline{H}$ be the genus 3 handlebody obtained by removing from H a neighborhood of the arc in which the axis of symmetry intersects one of the 0-handles. It is easy to see that in $\overline{H}$ the collection $\lambda_1, \lambda_2, \lambda_3 \subset \partial\overline{H}$ is a complete set of primitive curves, that is a primitive 3-tuple.

To construct some Dehn-derived pairs of Heegaard splittings, begin with two genus 2 handlebodies A and B, on each of whose boundaries lie three disjoint simple closed curves corresponding to $\lambda_1, \lambda_2, \lambda_3 \subset \partial H$. Let $\lambda_{ia} \subset \partial A$ (resp $\lambda_{ib} \subset \partial B$) be the curve corresponding to λ_i in A (resp B), for each $1 \leq i \leq 3$. Adopting (for comparison purposes) notation from [RS, Section 4.2], let $\alpha_{na}, \alpha_{sa}, \rho_a \subset A$ be the triple of curves obtained by pushing $\lambda_{1a}, \lambda_{2a}, \lambda_{3a}$ into the interior of A and let $\alpha_{nb}, \alpha_{sb}, \rho_b \subset B$ be the triple of curves obtained by pushing $\lambda_{1b}, \lambda_{2b}, \lambda_{3b}$ into the interior of B. Let N be a manifold constructed by identifying

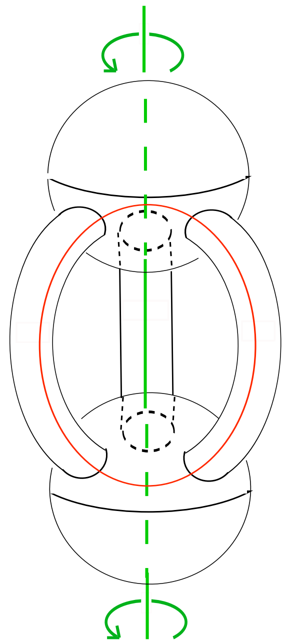

FIGURE 1

an annular neighborhood of λ_{1a} in ∂A with an annular neighborhood of λ_{1b} in ∂B and an annular neighborhood of λ_{2a} in ∂A with an annular neighborhood of λ_{2b} in ∂B. (After the identification, call the annuli $\mathcal{A}_n$ and $\mathcal{A}_s$ with core curves λ_1, λ_2 respectively .) Then identify the two 4-punctured spheres $\partial A - (\mathcal{A}_n \cup \mathcal{A}_s)$ and $\partial B - (\mathcal{A}_n \cup \mathcal{A}_s)$ by any homeomorphism.

This construction defines a genus 2 Heegaard structure on N, of course, but it also defines a genus 2 Heegaard splitting on M_0, the manifold obtained from N by arbitrary Dehn surgery on just the two curves $\rho_a \subset A$ and $\rho_b \subset B$, for surgery on these pushed-in primitive curves leaves A and B still handlebodies, handlebodies which we denote respectively H_a and H_b. What's more, the curves λ_1, λ_2 are each primitive in both H_a and H_b (though they are not necessarily a primitive pair in either). Thus the Heegaard splitting $M_0 = H_a \cup H_b$ gives rise to two potentially different genus 2 Heegaard structures on any manifold M_H that is obtained by simultaneously doing further Dehn surgery on the two curves λ_1, λ_2. That is, a manifold M_H obtained by arbitrary Dehn surgery on all four curves $\lambda_1, \lambda_2, \rho_a, \rho_b \subset N$ has two possibly distinct genus 2 Heegaard splittings, Dehn derived from the Heegaard splitting $M_0 = H_a \cup H_b$. One Heegaard structure $M_H = A_1 \cup B_2$ is obtained by pushing λ_1 to $\alpha_{na} \subset \mathrm{int}A$ and λ_2 to $\alpha_{sb} \subset \mathrm{int}B$ before doing Dehn surgeries on the four curves; the other $M_H = A_2 \cup B_1$ is obtained by pushing λ_1 to $\alpha_{nb} \subset \text{int}B$ and λ_2 to $\alpha_{sa} \subset \text{int}A$ before doing the Dehn surgeries. In each case, exactly two of the four Dehn surgered curves lie in each handlebody A and B before the Dehn surgery, and in that handlebody are a pushed-in primitive pair.

Proposition 3.1. *The two Heegaard splittings $A_1 \cup B_2$ and $A_2 \cup B_1$ of M_H become isotopic after at most a single stabilization.*

Proof. Let $\overline{A}$ and $\overline{B}$ be the genus 3 handlebodies derived from A and B respectively, just as $\overline{H}$ was derived from H. Here is a natural genus 3 Heegaard splitting of M_H: in contrast to the construction above, push *both* λ_1 to α_{na} and λ_2 to α_{sa}, so both curves (as well as ρ_a) lie in A before doing the Dehn surgeries. Although A may no longer be a handlebody after the Dehn surgeries, it follows from the discussion above that the result on $\overline{A}$ of the surgery on the three curves $\alpha_{na}, \alpha_{sa}, \rho_a \subset \overline{A} \subset A$ is still a genus three handlebody $\overline{A}'$. The complement of $\overline{A}'$ in M_H is also a handlebody B': a single 1-handle is added to B and surgery is done on the single curve $\rho_b \subset B$. Thus $M_H = \overline{A}' \cup B'$ is a genus 3 Heegaard splitting of M_H.

It's fairly easy to see that this Heegaard splitting is a stabilization of $A_1 \cup B_2$ (and so, symmetrically, $A_2 \cup B_1$). Indeed, an alternate way to construct $\overline{A}' \cup B'$ is to begin with $A_1 \cup B_2$ and add to A_1 (and so subtract from B_2) a regular neighborhood of the curve $\alpha_{sb} \subset \text{int}(B)$ and a straight arc from ∂B to α_{sb}. From this point of view, the inclusion $B' \subset B_2$ defines a genus 3 Heegaard splitting of the genus 2 handlebody B_2, and any such Heegaard splitting is necessarily stabilized (see [ST, Lemma 2.7]). The pair of stabilizing disks are also a pair of stabilizing disks for $\overline{A}' \cup B'$ □

4. A second construction, and a hybrid

Here is another natural, but less naive, way to find disjoint pairs of primitive curves on the boundary of a genus 2-handlebody and so to create a Dehn-derived pair of Heegaard splittings. Let F denote a torus with the interior of a disk removed. Then $F \times I$ is a genus 2-handlebody. For γ any properly embedded essential simple closed curve in F, $\gamma \times \{0\}$ (or symmetrically $\gamma \times \{1\}$) is a primitive curve in the handlebody $F \times I$. Indeed, for δ a properly embedded arc in F intersecting γ once, $\delta \times I$ is a meridian disk in $F \times I$ that intersects $\gamma \times \{0\}$ exactly once.

Following this observation, and the example of the previous section, here is a recipe for constructing candidate 3-manifolds. Begin with two copies A and B of the surface F and choose two essential (not necessarily disjoint) simple closed curves $\alpha_0, \alpha_1 \subset A$ and two essential (not necessarily disjoint) simple closed curves $\beta_0, \beta_1 \subset B$. Let $\lambda_{0a} = \alpha_0 \times \{0\} \subset \partial(A \times I), \lambda_{1a} = \alpha_1 \times \{1\} \subset \partial(A \times I), \lambda_{0b} = \beta_0 \times \{0\} \subset \partial(B \times I), \lambda_{1b} = \beta_1 \times \{1\} \subset \partial(B \times I)$. Identify an annular neighborhood of λ_{0a} in $A \times \{0\}$ with an annular neighborhood of λ_{0b} in $B \times \{0\}$ and call the core curve of the resulting annulus λ_0. Similarly identify an annular neighborhood of λ_{1a} in $A \times \{1\}$ with an annular neighborhood of λ_{1b} in $B \times \{1\}$ and call the core curve of the resulting annulus λ_1. Complete the identification of $\partial(A \times I)$ with $\partial(B \times I)$ along the remaining 4-punctured sphere arbitrarily. Call the resulting closed 3-manifold M_0, with Heegaard splitting $M_0 = (A \times I) \cup (B \times I)$.

The 3-manifold $M_{\times I}$ obtained from M_0 by doing arbitrary Dehn surgeries to the simple closed curves λ_0 and λ_1 has a Dehn-derived pair of Heegaard splittings: one comes from first pushing λ_0 into $A \times I$ and λ_1 into $B \times I$ before the Dehn surgery, the other comes from first pushing λ_1 into $A \times I$ and λ_0 into $B \times I$ before the Dehn surgery.

Remarks on stabilization It is not apparent to us that a single stabilization will make the two Dehn-derived splittings of $M_{\times I}$ equivalent. The argument of Proposition 3.1 does not immediately carry over: if both curves λ_{0a} and λ_{1a} are pushed into $A \times I$ there is no apparent arc so that the complement $\overline{A \times I}$ of a neighborhood of the arc in $A \times I$ is a genus 3 handlebody after an arbitrary Dehn surgery on the pushed in λ_{0a} and λ_{1a}. If there is a proper arc γ in A that intersects both curves $\alpha_0 \subset A$ and $\alpha_1 \subset A$ in a single point, then the complement $\overline{A \times I}$ after pushing the interior of γ into $A \times I$ is a genus 3 handlebody, and so a single stabilization suffices, but having such an arc γ is not the general situation. (What is required for such an arc γ to exist is that the slopes of α_0 α_1 in A are a distance at most two apart in the Farey graph [Mi, Figure 1]. In that case γ has the slope that is incident to the slopes of both α_0 and α_1 in the Farey graph.)

On the other hand, it is relatively easy to show that two stabilizations suffice to make the two splittings equivalent. To see this, push both λ_0 and λ_1 into $A \times I$ and connect them to respectively $A \times \{0\}$ and $A \times \{1\}$ by straight arcs. Then add a regular neighborhood of the arcs and of the pushed in curves λ_0 and λ_1 to $B \times I$ to create a genus 4 handlebody $\overline{B \times I}$ and simultaneously subtract the regular neighborhood from $A \times I$ to get the genus 4 handlebody $\overline{A \times I}$. The resulting genus 4 Heegaard splitting $M_0 = \overline{A \times I} \cup \overline{B \times I}$ becomes a Heegaard splitting $H_a^+ \cup H_b^+$ of $M_{\times I}$ after the prescribed Dehn surgery on λ_0 and λ_1. Using the argument of Proposition 3.1 it is easy to see that the Heegaard splitting $H_a^+ \cup H_b^+$ destablizes to the genus 3 splitting obtained by instead pushing λ_0 into $B \times I$ and then adding to $B \times I$ a regular neighborhood of $\lambda_1 \subset (A \times I)$ and a straight arc attaching it to $A \times \{1\}$. The argument of Proposition 3.1 applied again shows that this Heegaard splitting destabilizes to the genus 2 splitting in which λ_0 is pushed into $B \times I$ and λ_1 into $A \times I$, one of the Dehn-derived splittings. But this destabilization process is clearly symmetric: we could equally well have destabilized to the other genus 2 splitting, in which λ_0 is pushed into $A \times I$ and λ_1 into $B \times I$, and this is the other Dehn-derived splitting.

A further, call it a *hybrid* example of a Dehn-derived pair of splittings comes by combining the two constructions above: Identify annular neighborhoods of $\lambda_1, \lambda_2 \subset \partial H$ from Section 3 with annular neighborhoods of $\lambda_{0b}, \lambda_{1b} \subset \partial(B \times I)$ and identify the rest of ∂H with the rest of $\partial(B \times I)$ in any way. This gives a closed 3-manifold N with a Heegaard splitting $H \cup (B \times I)$. Let M_0 be a 3-manifold obtained by doing an arbitrary Dehn surgery on $\lambda_3 \subset \partial H$, after pushing it into $\mathrm{int}(H)$. Then M_0 has the genus 2 Heegaard splitting (exploiting the notation used above) $M_0 = H_a \cup (B \times I)$. Let M_{hybrid} be a closed 3-manifold obtained from M_0 by arbitrary Dehn surgeries on the two remaining curves $\lambda_1, \lambda_2 \subset \partial H_a \subset M_0$. The Dehn-derived pair of Heegaard splittings for M_{hybrid} is obtained by alternatively pushing λ_1 into H_a and λ_2 into $B \times I$ or vice versa. A single stabilization suffices to make the two splittings equivalent, essentially by the same argument as for M_H, in Proposition 3.1.

5. Filling the gap in [RS]

The gap in [RS] arises because of a faulty sentence in the midst of a long and technical argument which would be difficult to summarize. We see no good alternative to simply jumping into that argument at a reasonable breaking point and inserting the argument we now believe to be complete. The jumping in point is on p. 533, in the midst of trying to prove that all cases of multiple genus 2 Heegaard splittings have been covered in the earlier examples listed in that paper. Here M is a closed hyperbolike 3-manifold with Heegaard splittings $M = A \cup_P B = X \cup_Q Y$. The two splitting surfaces P and Q have been made to coincide on sub-surfaces $P_0 \subset P$ and $Q_0 \subset Q$. Then $P - P_0$ consists of annular components P_X and P_Y properly embedded in the handlebodies X and Y respectively, and $Q - Q_0$ consists of annular components Q_A, Q_B properly embedded in the handlebodies A, B respectively. With this as background, we now re-enter the proof of [RS, Theorem 9.4, Case 2] on page 533, at first echoing what is written there as the proof of Subcase A below. In filling the gap in the argument we also broaden the possible outcome, as expressed in Proposition 5.1:

Case 2: P_X and P_Y are both non-empty and the end of each curve in $\partial P_X \cup \partial P_Y$ is parallel to one of c_1 or c_2.

Proposition 5.1. *In this case, either*

I) *the splittings* $M = A \cup_P B = X \cup_Q Y$ *are related as in* [RS, Example 4.4] *or*
II) *the two splittings* $M = A \cup_P B = X \cup_Q Y$ *are Dehn-derived from a single genus* 2 *Heegaard splitting of another manifold* M_0.

Proof. If at least one annulus in each of P_X or P_Y is non-separating, then together they would give a non-separating, hence essential, torus in M. This contradicts our assumption that M is hyperbolike. So we may as well assume that each annulus in P_Y is separating. Hence the ends of P_Y are twisted in Y (see [RS, Definition 5.4]). No end of P_Y can also be twisted in X, for the union along the curve of the solid tori (one in X, one in Y) on which the curve is a torus knot would be a Seifert submanifold of M, contradicting the assumption that M is hyperbolike.

Subcase A: Some end of P_Y is parallel to an end of P_X. [The gap in [RS] was to view this as the only possibility.]

In this case, by [RS, Lemma 5.6] all of P_X is a collection of parallel non-separating longitudinal annuli in X. If P_Y has ends at both c_1 and c_2 then neither curve can be twisted in X. In this case each annulus in P_X is non-separating and so has ends that are non-parallel in Q. This implies that each annulus in P_X has one end at c_1 and one at c_2. Attach such an annulus in X to the tori in Y on which the c_i are twisted. The boundary of the thickened result would exhibit a Seifert manifold in M, again contradicting the assumption that M is hyperbolike. We conclude that P_Y has ends only at c_2, say.

If there were three or more annuli in P_Y (hence six or more ends of ∂P_Y at c_2) then there would be at least four ends of P_X at c_2. No annulus in P_X could have both ends at c_2 (since c_2 is not twisted in X) so there would also be at least four ends of P_X at c_1. This would contradict [RS, Lemma 9.5]. So we conclude that P_Y is made up of one or two annuli. If it's two annuli, necessarily separating and parallel in

Y, then, again by [RS, Lemma 9.5] some annulus in P_X has an end at c_2. It cannot have both ends at c_2 and must be non-separating and longitudinal in X, since c_2 is not twisted in X. In this case the relation between P and Q can be seen as follows (See [RS, Figure 32]): In [RS, Example 4.4, Variation 2], let P be the splitting given there with Dehn surgery curve in μ_{a_+} and Q be the same splitting given there but with Dehn surgery curve in μ_{a_-}. To view these simultaneously as splittings of the same manifold M, of course, the Dehn surgery curve has to be moved from μ_{a_+} to μ_{a_-}, dragging some annuli along, until the splitting surfaces P and Q intersect as described.

Suppose then that P_Y is a single annulus. It may have both ends on P_0 or it may have one end on P_0 and one end on an end of P_X. (If both ends of P_Y were also ends of P_X then these, together with ends of P_X at ∂P_0 parallel to c_2 would exhibit more than two annuli in P_X, hence more than two ends of P_X at c_1, contradicting [RS, Lemma 9.5].) If P_Y has one end on P_0 and one end on an end of P_X, the initial splitting by Q is as in [RS, Example 4.4, Variation 1] ($X = A_- \cup \sigma$), with a Dehn surgery curve lying in μ_{b_+}, say. If the splitting is altered by first putting the Dehn surgery curve in μ_{a_+} (yielding the same manifold M), then altering as in [RS, Example 4.4] (i. e. considering $A \cup_P B$ where $B = B_- \cup \sigma$) and then dragging the Dehn surgery curve from μ_{a_+} to μ_{b_+}, pushing before it an annulus from the 4-punctured sphere along which A_- and B_- are identified, we get the splitting surface P, intersecting Q as required. (See [RS, Figure 33].) This completes the proof that I) holds in Subcase A.

Subcase B: No end of P_Y is parallel to an end of P_X.

In view of [RS, Lemma 9.5], in this subcase P_Y and P_X each consist of exactly one separating annulus, P_Y twisted in Y with boundary curves parallel to c_2 (say) and P_X twisted in X with boundary curves parallel to c_1. This case is symmetric: the annulus in Q lying between the ends of P_Y is Q_A (say) and the annulus in Q lying between the ends of P_X is exactly Q_B. The annulus P_Y cannot be parallel to the annulus Q_A (else P_0 and Q_0 could be extended to include both) but rather the region between them is a solid torus $W_2 = A \cap Y$ on whose boundary the cores of the annuli are torus knots. Similarly $B \cap X$ is a solid torus W_1 on whose boundary the cores of the annuli Q_B and P_X are torus knots. The annulus P_Y ∂-compresses in Y to become a separating disk; it follows that $Y - W_2 = B \cap Y$ is a genus 2 handlebody H_{BY} on which the core a_2 of the annulus P_Y is primitive. Symmetrically, the curve c_1 (viewed as the core of the annulus Q_B) is primitive in H_{BY}, the curve

c_2 is primitive in the genus 2 handlebody $H_{AX} = X - W_1 = X \cap A$, as is the core curve a_1 of P_X.

Here is another way to describe the manifold M above: begin with the two genus 2 handlebodies H_{AX} (which contains disjoint primitive simple closed curves a_1, c_2 on its boundary) and H_{BY} (which contains disjoint primitive simple closed curves a_2, c_1 on its boundary). Construct a closed 3-manifold M_0 by identifying ∂H_{AX} to ∂H_{BY} by a homeomorphism that identifies a_i with c_i, $i = 1, 2$. Call the resulting curves α_1, α_2. Now recover M from M_0 by removing a tubular neighborhood of each α_i and replacing with the solid torus W_i; equivalently, do an appropriate surgery on each α_i in M_0. The two Heegaard splittings of M are then seen as follows: if α_1 is pushed into H_{AX} and α_2 into H_{BY} before the surgery on the curves, then the resulting Heegaard splitting is $M = X \cup_Q Y$; if α_1 is pushed into H_{BY} and α_2 into H_{AX} before the surgery then the resulting splitting is $M = A \cup_P B$. That is, the splittings of M are both Dehn-derived from the Heegaard splitting $M_0 = H_{AX} \cup H_{BY}$. Thus II) holds in Subcase B. □

6. A taxonomy of Dehn-derived splittings

Sections 3 and 4 give concrete examples of pairs of Dehn-derived fillings. In this section we show that these examples in fact constitute all pairs of Dehn-derived splittings. The argument exploits Berge's classification of pairs of primitive curves on genus 2 handlebodies [Be], though the classification here is slightly different.

Let H be a genus 2 handlebody, with $\lambda_1, \lambda_2, \lambda_3 \subset \partial H$ the disjoint simple closed curves described in Section 3. Denote by ρ the curve in the interior of H obtained by pushing λ_3 into H and let H_{surg} denote the handlebody obtained from H by a specified Dehn surgery on $\rho \subset \mathrm{int}(H)$. As in Section 4, let F denote a torus with the interior of a disk removed.

Proposition 6.1 (Berge)**.** *Suppose α and β are disjoint non-parallel primitive curves on the boundary of a genus 2 handlebody H. Then either*

A) *there is a Dehn surgery on $\rho \subset H$ and a homeomorphism $h : H \to H_{surg}$ so that $h(\alpha) = \lambda_1 \subset \partial H_{surg}$ and $h(\beta) = \lambda_2 \subset \partial H_{surg}$ or*
B) *there is a homeomorphism $h : H \to F \times I$ so that $h(\alpha) \subset F \times \{0\}$ and $h(\beta) \subset F \times \{1\}$.*

Proof. This classification is a variant of that described in [Be]. The Type II pair there, as well as some pairs of Type I, are exactly as described in alternative B). The interest is in the third example of a Type I pair, in [Be, Lemma 3.8 (3) via Figure 3]. In that example, H is viewed as divided into two solid tori by a separating disk D; let λ_a and λ_b be longitudes of the two solid tori into which D divides H. Then β is parallel to λ_b, and α is the band sum, via a band that crosses D once, of λ_b with a torus knot on the solid torus containing λ_a. This picture is equivalent to letting α be the band sum $\lambda_a \# \lambda_b$ (through D) of λ_b with λ_a, and then performing a Dehn surgery on a disjoint copy of λ_a that has been pushed into H, to become a core of the solid torus on which λ_a lies. Now translate: relabel $\lambda_b \subset \partial H$ as λ_2 and $\lambda_a \subset \partial H$ as λ_3. Then $\lambda_a \# \lambda_b$ corresponds to λ_1. The construction just described is then to push λ_3 into the interior of H and perform some surgery to get H_{surg}. Afterwards α corresponds to $\lambda_1 \subset \partial H_{surg}$ and β corresponds to $\lambda_2 \subset \partial H_{surg}$. This is exactly alternative A). □

Following Propositions 5.1 and 6.1 there is a fairly clear description of the cases of multiple Heegaard splittings that are missing from [RS]. According to Proposition 5.1 the only missing cases are pairs of splittings that are Dehn-derived from an initial splitting $H_{AX} \cup H_{BY}$ of a manifold M_0. First determine which of alternatives A) and B) apply to the pairs of surgery curves as they lie on the boundaries of the respective handlebodies: $\{a_1, c_2\} \subset H_{AX}$ or $\{a_2, c_1\} \subset H_{BY}$. If both are of type A) then the pair of splittings is Dehn-derived as in the construction of M_H in Section 3. If both are of type B) then the pair of splittings is Dehn-derived as in the construction of $M_{\times I}$ in Section 4. If one is of type A) and one of type B) then the pair of splittings is Dehn-derived as in the construction of M_{hybrid} in Section 4.

It is worth mentioning that there is another view of a pair of primitive curves lying on a handlebody as in A) of Proposition 6.1, a view that more closely resembles that in B): Let α, β, γ be simple closed curves in F so that each pair of curves intersects in exactly one point. (For example, choose curves in F of slopes $0, 1, \infty$.) Then it is fairly easy to see that the three curves $\alpha \times \{0\}, \beta \times \{1\}, \gamma \times \{\frac{1}{2}\}$ lie in the handlebody $F \times I$ just as $\lambda_1, \lambda_2, \rho$ lie in H in the description preceding Proposition 6.1. So the primitive curves in description A) can be made to look like a special case of those in description B), but with the cost that an extra Dehn surgery has to be performed on a specific curve in the interior of $F \times I$. This is the twisted product view of [Be, 3.2].

7. Distance

It would seem possible that the Dehn-derived pairs of Heegaard splittings exhibited above could coincidentally all be contained among the examples already listed in [RS], for there is no claim that the types of examples of multiple Heegaard splittings we have offered here and in [RS] do not overlap. But in fact there is an invariant which does show that at least some Dehn-derived pairs of Heegaard splittings described above did not already occur in a different guise in [RS]. This invariant had not yet been introduced when [RS] was written and is called the *(Hempel) distance* of the Heegaard splitting [He]. We briefly review:

Definition 7.1. *A Heegaard splitting $H_1 \cup_S H_2$ has Hempel distance at most n if there is a sequence $c_0, ..., c_n$ of essential simple closed curves in the splitting surface S so that*

- *for each $i = 1, ..., n$, $c_i \cap c_{i-1} = \emptyset$*
- *c_0 bounds a disk in H_1*
- *c_n bounds a disk in H_2*

If the splitting has distance $\leq n$ but not $\leq n-1$, then the distance $d(S) = n$.

A Heegaard splitting of distance 0 is called *reducible*; one of distance ≤ 1 is called *weakly reducible.* Any Heegaard splitting of a reducible manifold is reducible. A Heegaard splitting of distance ≤ 2 is said to have the *disjoint curve property* [Th]; any Heegaard splitting of a toroidal 3-manifold has the disjoint curve property ([He], [Th]). A weakly reducible genus 2 Heegaard splitting is also reducible, so an irreducible Heegaard splitting of genus 2 has distance at least 2 ([Th]).

In the other direction we have:

Proposition 7.2. *Suppose the manifold M has a Dehn-derived pair of Heegaard splittings. Then each of these Heegaard splittings has Hempel distance at most* 3.

Proof. Suppose the splittings are Dehn-derived from a splitting $M_0 = H_a \cup_S H_b$ via the disjoint pair of simple closed curves $\lambda_1, \lambda_2 \subset S$. With no loss of generality, consider the splitting $M = A \cup_S B$ obtained by pushing λ_1 into $\text{int}(H_a)$ and λ_2 into $\text{int}(H_b)$ before doing Dehn surgery on the λ_i. Since λ_1 is primitive in H_a there is a properly embedded essential disk $D_a \subset H_a$ that is disjoint from λ_1. (For example D_a can be obtained from a meridian disk $D_1 \subset H_a$ that intersects λ_1 in a single point by band-summing together two copies of D_1 along a subarc of $\lambda_1 - D_1$.) D_a is then also disjoint from the curve $\alpha_1 \subset H_a$ obtained by pushing λ_1 into $\text{int}(H_a)$, so D_a remains intact as a meridian of A after surgery on α_1. Hence ∂D_a and λ_1 are disjoint curves in ∂A.

Symmetrically, there is a meridian $D_b \subset B$ so that ∂D_b and λ_2 are disjoint curves in ∂B. Then the sequence $\partial D_a, \lambda_1, \lambda_2, \partial D_b$ shows that the splitting $A \cup_S B$ has distance at most 3. □

Proposition 7.3. *All examples of multiple Heegaard splittings appearing in* [RS, Section 4] *have Hempel distance* ≤ 2.

Proof. Following the comments above we can restrict attention to irreducible, atoroidal manifolds. We briefly run through the examples as they appear in [RS, Section 4]. Typically the description of an example $H_1 \cup_S H_2$ in [RS] consists of two parts: A collection of annuli $\mathcal{A} \subset S$ along which ∂H_1 and ∂H_2 are identified, followed by an arbitrary identification of $\partial H_1 - \mathcal{A}$ with $\partial H_2 - \mathcal{A}$. From this point of view the simple closed curves $\partial \mathcal{A} \subset S$ that separate one sort of region from the other will be called the *seams* of the Heegaard splitting. We will observe that in [RS] some seam is always disjoint from an essential disk in H_1 and an essential disk in H_2. This demonstrates that the splitting has the disjoint curve property and so has distance ≤ 2.

To be specific: In [RS, Subsection 4.1], [RS, Subsection 4.2, Variation 1] and [RS, Subsection 4.4, Variations 1 and 2], the meridians of the 1-handles e_a and e_b are disjoint from the seams. [RS, Subsection 4.2, Variation 2] is slightly more complicated. It is a bit like the construction in Section 3 above: Handlebodies A and B are identified along neighborhoods of all three curves $\lambda_i, i = 1, 2, 3$, Dehn surgery is done to all three, with λ_1, λ_2 pushed into A and λ_3 into B (then vice versa). But there is a meridian of A disjoint from λ_1 and λ_2 and a meridian of B disjoint from λ_1 and λ_3, so a seam parallel to λ_1 demonstrates that the splitting of [RS, Subsection 4.2, Variation 2] has the disjoint curve property.

The manifolds in [RS, Subsection 4.3] and [RS, Subsection 4.4, Variations 3, 4, and 7] are all toroidal, so they are of distance ≤ 2. What remains are [RS, Subsection 4.4, Variations 5 and 6] and we adopt the terminology there. In Variation 5, with, say, $\rho_a \subset A_-$, the seams that are the boundary of the 4-punctured sphere $\partial A_- \cap \partial \Gamma$ are all disjoint from the meridian of the 1-handle $e_b \subset B$ and, in A_-, any one of these seams together with ρ_a lie in A_- as two of the λ_i's of Section 3 above lie in H. In particular, there is a meridian of A_- disjoint from both the seam and from ρ_a. Thus that seam again illustrates that the splitting has the disjoint curve property.

The argument for Variation 6 is much the same. First note that if, in that Variation, Dehn surgeries are done on two curves parallel to σ, then the resulting manifold has a Seifert piece and so has distance ≤ 2. So the only change we need to consider from Variation 5 is Dehn surgery on a single curve parallel to σ. If that curve lies in B the argument for Variation 5 suffices; if it is in A_- this merely forces us to pick a specific seam in the argument for Variation 5, a seam parallel to the new surgery curve. □

In contrast, some of the examples constructed in this paper can be shown to have distance 3, so they cannot have appeared in any case considered in [RS]. See [Be2] for details.

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