NON-PL IMBEDDINGS OF 3-MANIFOLDS.

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1. Introduction. All imbeddings will be locally flat and all isotopies will be ambient isotopies. Let M^m, N^{m+2} be closed PL manifolds, $m \geqslant 3$. For any imbedding $f: M^m \to N^{m+2}$ there is an obstruction in $H^3(M; Z_2)$ to isotoping f to a PL imbedding [6]. However, it follows from the twisted product structure theorem of [9] that for $m \geqslant 5$ there is a PL manifold M' and a homeomorphism $g: M' \to M$ such that $f \circ g: M' \to N$ is isotopic to a PL imbedding. Thus for $m \geqslant 5$ there is a natural way of replacing any imbedding of M in N with a PL imbedding of a homeomorphic manifold M'.

Here we study the analogous situation for $m = 3$. Since PL structures on 3-manifolds are unique, the above theorem does not extend directly. However, we show there is a biunique correspondence between isotopy classes of non-PL imbeddings of M in N and isotopy classes of PL imbeddings (with an appropriate condition on fundamental group) of a manifold M' homology equivalent to M.

In particular, we will show in Section 4 that the cobordism classification of non-PL 3-knots in S^5 [4, 2] is equivalent to the cobordism classification of PL imbeddings of certain homology 3-spheres in S^5. The correspondence is natural in the sense that a 3-knot and the corresponding imbedded homology sphere have (non-properly) homotopy equivalent complements in S^5, and the Seifert surface for the homology 3-sphere has a Seifert pairing isomorphic to that which Cappell and Shaneson define for the knot.

Explicitly the main theorem is as follows. Let H be a homology sphere bounding an index 8 PL parallelizable 4-manifold. Let M and N be closed PL 3 and 5 dimensional manifolds respectively. Let $\mathrm{Imb}_{\not{PL}}(M,N)$ denote isotopy classes of imbeddings $M \to N$ which do not contain a PL imbedding, and let $\mathrm{Imb}_H(M \# H, N)$ denote isotopy classes of PL imbeddings $g: M \# H \to N$ with the property that the natural map $i_{\#}: \pi_1(H) \to \pi_1(N - g(M \# H))$ is trivial. Here $i_{\#}$ is induced by inclusion after isotoping $H - \mathring{D}^3 \subset M \# H$ off of $g(M \# H)$ in N.

Manuscript received January 12, 1976.

0002-9327/78/1003-0539 $01.50

THEOREM 1.1. *There is a natural* 1-1 *correspondence* $\theta: \mathrm{Imb}_{PL}(M,N) \leftrightarrow \mathrm{Imb}_H(M \# H, N)$ *such that for* $f \in \mathrm{Imb}_{PL}(M,N)$, $N - f(M)$ *and* $N - \theta(f)(M \# H)$ *are homotopy equivalent (open) manifolds.*

Remarks. For all known examples of H there will not be a proper homotopy equivalence, since if $\pi_1(H) \neq 0$, the ends will have different fundamental groups.

An obvious relative version for $\partial M \neq \varnothing \neq \partial N$ is also true; the proof is identical to that below.

The requirement that $i_{\#}: \pi_1(H) \to \pi_1(N - g(H))$ be trivial in the definition of $\mathrm{Imb}_H(H,N)$ is necessary even when N is S^5. In fact there is a homology 3-sphere H and a smooth imbedding $H \hookrightarrow S^5$ such that the natural inclusion $\pi_1(H) \to \pi_1(S^5 - H)$ is non-trivial. The construction and its implications for a more general problem will appear elsewhere.

2. Preliminary Lemmas. Let H be a homology sphere bounding a PL parallelizable manifold of index 8.

LEMMA 2.1. $H \times S^1$ *bounds a unique topological manifold* U *homotopy equivalent to the circle, but does not bound such a PL manifold.*

Proof. That U exists is implicit in [1,11]. A detailed proof for H the dodecahedral space appears in [7,8]. This proof readily extends to any such H.

If U and U' are too such manifolds, then $U \cup_\partial U'$ is a homotopy $S^4 \times S^1$, and hence is homeomorphic to $S^4 \times S^1$ [10], and hence bounds $D^5 \times S^1$. The inclusions $U, U' \to D^5 \times S^1$ are homotopy equivalences, so by the TOP s-cobordism theorem $U \cong U'$.

We will consider H as the union along the boundary of $H_- \cong H - \mathring{D}^3$ and D^3. The natural imbeddings of $D^3 \times S^1$ and $H_- \times S^1$ in ∂U will be denoted $(D^3 \times S^1)_U, (H_- \times S^1)_U$.

LEMMA 2.2. *The unknotted imbedding* $S^2 \to S^4$ *extends to a locally flat imbedding* $H_- \to D^5$ *such that the complement of an open tubular neighborhood of* H_- *in* D^5 *is* U. *The imbedding is unique up to isotopy rel boundary and is not isotopic to a PL imbedding.*

Proof. Attach $H_- \times D^2$ to U along $(H_- \times S^1)_U$. The result is a contractible 5-manifold with boundary S^4, and hence D^5. The composition $H_- \times \{0\} \hookrightarrow H_- \times D^2 \hookrightarrow D^5$ then defines the required imbedding.

If $i_1: H_- \hookrightarrow D^5$ and $i_2: H_- \hookrightarrow D^5$ are two such imbeddings, the uniqueness of U implies there is a PL homeomorphism $h: D^5 \to D^5$ such that $i_2 h = i_1$ and $h = $ identity near ∂D^5. But any such PL homeomorphism is isotopic to the identity by Alexander's trick.

Remark 2.3. We will make use of the following observation. Suppose a 5-manifold V is the union along the boundary of two 5-manifolds W_1 and W_2 (written $V = W_1 \cup_\partial W_2$). Let $x: S^1 \to \partial W_1 \cong \partial W_2$ be an imbedding of the circle in V, with trivial tubular neighborhood ν restricting to tubular neighborhoods ν_i of x in ∂W_i, $i = 1,2$. Then $W_i - \nu \cong W_i$, and $V - \nu$ is the union of W_1 and W_2 along $\partial W_1 - \nu_1$ and $\partial W_2 - \nu_2$.

Note that each ν_i defines a cross-section of the S^3 bundle $\dot{\nu}$. Since $\pi_1(S^3) = 0$, any two cross-sections are isotopic. Hence if $\partial D^2 \times S^3$ is attached to $V - \nu$ by a bundle map $h: \partial D^2 \times S^3 \to \dot{\nu}$, we may assume that for $S^3 = D_1^3 \cup_\partial D_2^3$, $h^{-1}(\nu_i) = \partial D^2 \times D_i^3$. Therefore if surgery is performed on x in V with a given framing, then there are framings for ν_1 and ν_2 in ∂W_1 and ∂W_2 with the following property: The manifold V' obtained by the surgery on V is the boundary union of the manifolds obtained from W_1 and W_2 by attaching 2-handles to ν_1 and ν_2 with these framings.

Let T be a simply connected manifold obtained from $H_- \times D^2$ by attaching r 2-handles along circles $x_1, \ldots, x_r$ in $H_- \times (q)$, $q \in \partial D^2$. Let Q be the manifold obtained from U by attaching r 2-handles to $x_1, \ldots, x_r$ in $(H_- \times (q))U$ with the same framings used to define T.

LEMMA 2.4. *There is a natural locally flat isotopy class of imbeddings* $i: (D^3, \partial D^3) \hookrightarrow (T, \partial H \times \{0\})$, *such that the complement of an open tubular neighborhood of* $i(D^3)$ *in* T *is homeomorphic to* Q.

Proof. Attach a 2-handle $L \cong D^3 \times D^2$ to U along $(D^3 \times S^1)_U$, and call the resulting manifold P. It is easily seen that P is contractible and, with the natural identification along the boundary, $P \cup_\partial (H_- \times D^2)$ is a homotopy S^5, and hence S^5. On $P \cup_\partial (H_- \times D^2)$ perform surgery on $x_1, \ldots, x_r$ in $H_- \times (q)$ with framings chosen as in Remark 2.3 so that the resulting PL manifold V will be $T \cup_\partial Q'$, where Q' is obtained from Q by attaching L along $(D^3 \times S^1)_U$.

Since V was obtained from S^5 by doing surgery on r-circles, it is the connected sum of r 3-sphere bundles over S^2 and so bounds W, the boundary sum of r 4-disk bundles over S^2. Since T is simply connected, the inclusions $Q' \hookrightarrow W$, $T \hookrightarrow W$ are homotopy equivalences. By the PL h-cobordism theorem, $Q' \overset{PL}{\cong} T$. Since Q' contains the required 3-disk $D^3 \times \{0\} \subset L$, so does T.

The PL homeomorphism $h: Q' \to T$ is well defined up to isotopy. Indeed, from the PL h-cobordism theorem there is a PL homeomorphism $H: W \to T \times I$ such that $H | T \to T \times \{0\}$ is the identity. If $H': W \to T \times I$ is any other such PL homeomorphism, then $H' \cdot H^{-1}: T \times I \to T \times I$ is a PL pseudoisotopy. Since $\pi_1(T) = 0$, $H' \cdot H^{-1}$ is PL isotopic rel boundary to an isotopy [5]. Hence $H | Q'$ is isotopic rel boundary to $H' | Q'$. This completes the proof of Lemma 2.4.

Remark 2.5. If s further 2-handles are attached to T to obtain T', the circles upon which they are attached will be null-homotopic, and hence null-isotopic in $\partial Q'$ and ∂T. Thus $T' \cong T \natural s(S^2 \times D^3)$, and the natural imbedding $D^3 \hookrightarrow T'$ which the above lemma would provide is also that obtained from the composition $(D^3 \hookrightarrow T \hookrightarrow T \natural s(S^2 \times D^3) \cong T'$.

3. Proof of Theorem 1.1.

Step 1. Definition of θ. Let $f: M \to N$ be an imbedding not isotopic to a PL imbedding, $D^3 \subset M$ be a disk PL imbedded in M, and $D^3 \times D^2$ be the restriction to D^3 of some normal bundle ν to M in N. $D^3 \times D^2$ is uniquely defined up to ambient isotopy.

Since the imbedding $M \to N$ is not isotopic to a PL imbedding, while N is PL, the Kirby-Siebenmann obstruction to isotoping the bundle PL structure ν_M on ν, inherited from the PL structure on M, to the PL structure ν_N inherited from N, is the non-trivial element α of $H^3(\nu;\, Z_2)$ [6].

Lemma 2.1 applied to $D^3 \times D^2$ provides a non-PL locally flat imbedding $M \# H \to \nu_M$, which we denote $\theta(f)$. Since $\theta(f)$ is not isotopic in ν_M to a PL imbedding, the obstruction to isotoping the PL structure ν_M to one in which $\theta(f)(M \# H)$ is PL imbedded is also α. Since $2\alpha = 0$, $\theta(f)$ is isotopic to an imbedding into N which is PL.

Step 2. Definition of $\psi: \mathrm{Imb}_H(M \# H, N) \to \mathrm{Imb}_{PL}(M, N)$. Let $g: M \# H \to N$ be in $\mathrm{Imb}_H(M \# H, N)$ and $H_- \times D^2$ be the restriction to H_- of some normal bundle ν to $g(M \# H)$ in N. Since H_- is acyclic, the trivialization $H_- \times D^2$ is well defined. For $q \in \partial D^2$, let $x_i: S^1 \to H \times (p)$, $i = 1, \ldots, r$, be disjoint circles imbedded in $H_- \times (q)$ representing $x_i \in \pi_1(H)$ such that $\pi_1(H)/[x_1, \ldots, x_r] = 0$. By general position and the definition of $\mathrm{Imb}_H(M \# H, N)$, $x_1, \ldots, x_r$ bound 2-disks in $N - \nu$. Attach 2-handles to $H_- \times D^2$ using these disks as core disks. Lemma 2.4 then provides an extension of $g(M_-)$ to an imbedding of M, which we call $\psi(g)$.

The map $\psi(g)$ is well defined, for the only real choice involved is that of the circles $x_1, \ldots, x_r$ and their spanning disks. Remark 2.5 shows that this choice is irrelevant.

That $\psi(g)$ is not PL follows much as did the proof that $\theta(f)$ is PL.

Step 3. $\psi\theta(f) = f$. Let $(H_-, \partial H_-) \to (D^3 \times D^2, \partial D^3 \times \{0\})$ be as defined in Lemma 2.2, and $H_- \times D^2$ be a tubular neighborhood of H_- in $D^3 \times D^2$. Choose disjoint imbedded circles in $H_- \times (q)$ which normally generate $\pi_1(H)$, choose disjoint disks in $(D^3 \times D^2) - (H_- \times D^2)$ bounding the circles, and attach 2-handles with these as core disks to $H_- \times D^2$ in $(D^3 \times D^2) - (H_- \times D^2)$, obtaining a manifold $T \subset D^3 \times D^2$. Let $i(D^3)$ be the imbedded D^3 given by Lemma 2.4. It suffices to show that $i(D^3)$ is isotopic in $D^3 \times D^2$ to $D^3 \times \{0\}$.

From the definitions in Lemmas 2.2 and 2.4 it is evident that the complement of a neighborhood of $h(D^3)$ in $D^3 \times D^2$ is simply the manifold obtained by identifying two copies of U along $(H_- \times S^1)_U$. This manifold is a homotopy circle bounded by $S^3 \times S^1$, and hence is $D^4 \times S^1$ [10]. Thus $h(D^3)$ is the standard unknotted 3-disk in $D^3 \times D^2$.

Step 4. $\theta\psi(g) = g$. It is evident from the definitions that θ and ψ commute with ambient PL homeomorphisms of N. That is, if $h: N \to N$ is a PL homeomorphism, then $\theta\psi(h \circ g) = h \circ \theta\psi(g)$. Given g, it therefore suffices to define a PL homeomorphism $h: N \to N$ such that $h \circ g = \theta\psi(g)$, for then $g = \theta\psi(h^{-1} \circ g) = \theta(\psi\theta)\psi(h^{-1} \circ g) = \theta\psi\big(\theta\psi(h^{-1} \circ g)\big) = \theta\psi(g)$ by Step 3.

Let T be the manifold of Lemma 2.4 as imbedded in N in Step 2. Let ν be an open tubular neighborhood of $g(H_- \times \{0\})$ in T, and ν' an open tubular neighborhood of $\theta\psi(g)(H_- \times \{0\})$ in T. It suffices to show that the natural homeomorphism of the boundaries of $T - \nu$ and $T - \nu'$ extends to a homeomorphism of $T - \nu$ to $T - \nu'$.

$T - \nu$ is the space obtained from $H_- \times \partial D^2 \times I$ by attaching 2-handles to circles $x_1, \ldots, x_r$ in $H_- \times (q) \times \{1\}$ which normally generate $\pi_1(H_-)$. On the other hand, $T - \nu'$ is obtained from $U \cup_{(D^3 \times S^1)_U} U$ by attaching 2-handles to one of the boundary components in the same manner. Thus, from Remark 2.3, it follows that the manifold V, defined as the union along the boundary of $T - \nu$ and $T - \nu'$, may be obtained by identifying two copies of U along their entire boundary and doing surgery along $x_1, \ldots, x_r$ in $H_- \times (q) \subset (H_- \times S^1)_U \subset U \cup_\partial U$.

But, as in Lemma 2.2, $U \cup_\partial U$ is simply $S^4 \times S^1$. Hence V is the connected sum of $S^4 \times S^1$ and r 3-sphere bundles over S^2. Thus V bounds the boundary sum W of $D^5 \times S^1$ and r 4-disk bundles over S^2, and the inclusions $T - \nu \to W$ and $T - \nu' \to W$ are homotopy equivalences. By the s-cobordism theorem, $T - \nu \cong T - \nu'$. This completes Step 4.

The proof of Theorem 1.1 is completed by observing that the inclusion $(D^3 \times S^1)_U \to U$ is a homotopy equivalence, so that $N - f(M)$ and $N - \theta(f)(M)$ are also homotopy equivalent.

4. Example: Non-PL Knots. Among the peculiarities which arise in the study of knots, one of the most spectacular is the family of non-PL locally flat 3-knots in S^5 [4]. These knots have been extensively studied by Cappell and Shaneson [2]. One difficulty which arises is the absence of the topological transversality theory at dimension 4 which is needed to construct a Seifert surface for the knot [3,8]. This is overcome in [2] by taking the product of the knot complement with CP(2), then using topological transversality at dimension 8 to define a "suspended" Seifert surface for the knot.

Theorem 1.1 provides an alternative but equivalent definition of the Seifert pairing of a non-PL knot $f: S^3 \to S^5$. If f is not isotopic to a PL imbedding, then $\theta(f): H \to S^5$ is isotopic to a PL imbedding and so has a Seifert surface V, with $\partial V \cong H$, and an associated Seifert pairing. We define the Seifert pairing of $f(S^3)$ to be that of V and show that this Seifert pairing is equivalent to that of [2].

First we outline the Cappell-Shaneson definition. For ν an open tubular neighborhood of $f(S^3)$, let $h: S^5 - \nu \to S^1$ be a homology equivalence extending a bundle trivialization $\dot{\nu} \to S^1$. Let q be a point in S^1 and $p: (S^5 - \nu) \times \mathrm{CP}(2) \to S^5 - \nu$ be the projection. Homotope $hp: (S^5 - \nu) \times \mathrm{CP}(2) \to S^1$ rel boundary so that hp is TOP transverse to q [3]. Then $(hp)^{-1}(q) = N$ is a suspended Seifert surface for $f(S^3)$, and the Seifert pairing is the linking pairing on the quotient of the kernel of $H_4(N) \to H_4(S^5 \times CP(2))$ by its torsion subgroup.

Let ν' be an open tubular neighborhood of $\theta(f)(H)$. By definition of $\theta(f)(H)$, $S^5 - \nu = (S^5 - \nu') \cup_\varphi (U)$, where $\varphi(H_- \times S^1)_U \to \nu'$ is a bundle map covering the inclusion $H_- \subset H$. By Siebenmann's splitting criterion [11] the manifold $U \times \mathrm{CP}(2)$, homotopy equivalent to cone$(H) \times S^1 \times \mathrm{CP}(2)$, is the product with S^1 of a manifold N', where $\partial N' = H \times \mathrm{CP}(2)$ and N' is homotopy equivalent to cone$(H) \times \mathrm{CP}(2)$. Clearly $N = (V \times CP(2)) \cup_{\varphi'} (N')$, is a suspended Seifert manifold for $f(S^3)$, where φ' is the natural homeomorphism of $H_- \times \mathrm{CP}(2) \subset \partial N'$ to $H_- \times \mathrm{CP}(2) \subset \partial V \times \mathrm{CP}(2)$. Since N' is homotopy equivalent to CP(2), the kernel of $H_4(N) \to H_4(S^5 \times CP(2))$ is precisely $H_4(V)$. Thus the Seifert pairing as defined by Cappell and Shaneson is precisely the Seifert pairing of V.

Finally we prove

LEMMA 4.1. *Any PL imbedding of H in S^5 is cobordant to an imbedding in* $\mathrm{Imb}_H(H, S^5)$.

COROLLARY 4.2. *θ induces a natural equivalence between cobordism classes of non-PL 3-knots and cobordism classes of PL imbeddings of H in S^5.*

Proof of Lemma 4.1. For a PL function $f: H \hookrightarrow S^5$, we must show that f is cobordant to an imbedding g such that $i_\#: \pi_1(H) \to \pi_1(S^5 - g(H))$ is trivial. Let $\nu: H_- \times D^2 \to S^5$ be a normal bundle trivialization extending $f|H_-$, and, letting $I = [0,1]$ be the radial coordinate of D^2, let $\nu_- = \nu(H_- \times \partial D^2 \times [\frac{1}{2}, 1])$. Then $\partial \nu_-$ is naturally homeomorphic to $\partial(U \cup_{(D^3 \times S^1)_U} U)$ and $(U \cup_{(D^3 \times S^1)_U} U) \cup_\partial \nu_- \cong U \cup_\partial U \cong S^4 \times S^1$. Regarding S^4 as ∂D^5, this defines an imbedding $h: \nu_- \to \partial D^5 \times S^1 \subset D^5 \times S^1$.

Attach $D^5 \times S^1$ to $S^5 \times I$ by identifying $\nu_- \times \{1\} \subset S^5 \times \{1\}$ with $h(\nu_-) \subset \partial D^5 \times S^1$, and call the resulting manifold W. Since the inclusion $h: \nu_- \to D^5 \times S^1$ is a homology equivalence, it follows easily that there is a PL homeomorphism $G: W \to S^5 \times I$ which is the identity on $S^5 \times \{0\} \subset W$.

The boundary components of W are $\partial_0 W = S^5 \times \{0\}$ and the manifold $\partial_1 W$ obtained from $S^5 \times \{1\}$ by removing ν_- and replacing it with $U \cup_{(D^3 \times S^1)_U} U$.

Since the inclusion induced homomorphism $\pi_1(H_-) \to \pi_1(U)$ is trivial, the imbedding $f: H \hookrightarrow \partial_1 W$ is in $\mathrm{Imb}_H(H, \partial_1 W)$. Thus the composition

$$H \times I \xrightarrow{f \times 1_I} S^5 \times I \subset W \xrightarrow{G} S^5 \times I$$

defines a cobordism from f to an element of $\mathrm{Imb}_H(H, S^5)$.

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