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Subgroups of $SL(2, R)$ Freely Generated by Three Parabolic Elements

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For m a real number, let $G(m)$ be the subgroup of $SL(2, R)$ generated by

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}.$$

In 1947 Sanov [S] proved that $G(2)$ is free. Some years later, Brenner [Br] showed that $G(m)$ is free for all $|m| \geqslant 2$ and Chang, Jennings and Ree [CJR] showed that values of m for which $G(m)$ is not even torsion free are dense in the interval $[-2, 2]$.

Recently, Bachmuth and Mochizuki [BM] defined subgroups $G(\alpha, \beta, \gamma)$ of $SL(2, R)$ generated by

$$h_0 = \begin{pmatrix} 1+\alpha & -\alpha \\ \alpha & 1-\alpha \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix},$$

showed that for $\alpha, \beta, \gamma > 4.45$ $G(\alpha, \beta, \gamma)$ is free, and asked whether $G(\alpha, \beta, \gamma)$ can be contained in any free subgroup of rank 2.

In Section 1 we present preliminary notions and quick geometric proofs of these theorems of Sanov, Brenner and Chang–Jennings–Ree. Much more is known about these groups $G(m)$ and other subgroups of $SL(2, R)$ of rank two than these results (see for example [N], [LU_1], [LU_2]). Our intention here is to recapitulate only that part which is geometrically obvious and which motivates our analysis of the groups $G(\alpha, \beta, \gamma)$.

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In Section 2 it is shown that the lower bound on α, β, γ for which $G(\alpha, \beta, \gamma)$ is free is 3 (see also [M]). More generally, if $1/|\alpha|+1/|\beta|+1/|\gamma| \leqslant 1$ then $G(\alpha, \beta, \gamma)$ is free, and, in the range $1/|\alpha|+1/|\beta|+1/|\gamma| \geqslant 1$, the values of α, β, γ for which $G(\alpha, \beta, \gamma)$ is not even torsion free are dense. The theory therefore parallels that of $G(m)$, with the interval $1/|m| < \frac{1}{2}$ replaced by the region $1/|\alpha|+1/|\beta|+1/|\gamma| < 1$.

In Section 3 we define an action of $PSL(2, Z)$ on the region $1/|\alpha|+1/|\beta|+1/|\gamma| \leqslant 1$ and show that $G(\alpha, \beta, \gamma)$ is conjugate to $G(\alpha', \beta', \gamma')$ if and only if (α, β, γ) is equivalent to $(\alpha', \beta', \gamma')$ under this action.

In Section 4 we show that if $1/|\alpha|+1/|\beta|+1/|\gamma| = 1$, and $2 \leqslant \beta \leqslant \alpha, \gamma$ then $G(\alpha, \beta, \gamma)$ is contained in a rank two free discrete subgroup of $SL(2, R)$ if and only if $(\alpha, \beta, \gamma) = (4, 2, 4)$. Thus there are infinitely many subgroups $G(\alpha, \beta, \gamma)$ which are free, yet contained in no discrete rank two subgroup of $SL(2, R)$.

Remarks Since changing the sign of α, β, γ does not change the group $G(\alpha, \beta, \gamma)$, we will assume throughout that $\alpha, \beta, \gamma > 0$.

Any three parabolic elements of $SL(2, R)$ with distinct fixed points may be conjugated to elements of the form h_0, h_1, h_2, by conjugating the three fixed points to $1, \infty, 0$.

I would like to thank the referee for his helpful comments and suggestions.

1. THE GEOMETRY

$SL(2, R)$ acts on the closed extended complex upper half-plane H as a group of linear fractional transformations. This action preserves angles and circles in the extended complex plane. The subgroup of $SL(2, R)$ which acts as the identity transformation is the center $\pm I$; the quotient $SL(2, R)/\pm I$ is denoted $PSL(2, R)$. The transformation $S(z) = (-z+i)/z+i$ in $GL(2, C)$ maps H onto the closed unit disk. Thus, after conjugation by S, $PSL(2, R)$ acts on the disk. It is convenient to use this model for $PSL(2, R)$ because it is compact and contains much symmetry. In particular, $e^{i\theta}$ in the unit disk model corresponds to $\tan \theta/2$, so points on the unit circle symmetric in the x-axis correspond to points in R which differ by a sign and points on the circle symmetric in the y-axis correspond to reciprocals in R.

In the figures below a real number x will be used as a label for the point $S(x)$ on the unit circle U; thus we are using D as a picture of H.

A bridge between geometry and algebra is given in the following well-known lemma. Since it is both simple and crucial, a proof is included.

1.1 LEMMA *Let $h_1, \ldots, h_n$ be parabolic transformations, h_i with fixed point p_i on U. Let each h_i map a point $a_i \neq p_i$ on U to a point b_i and let C_i be the* *arc of U between a_i and b_i that contains p_i. If the arcs C_i are disjoint, except possibly at their endpoints, then $h_1, \ldots, h_n$ are a basis for a free group* (Figure 1).

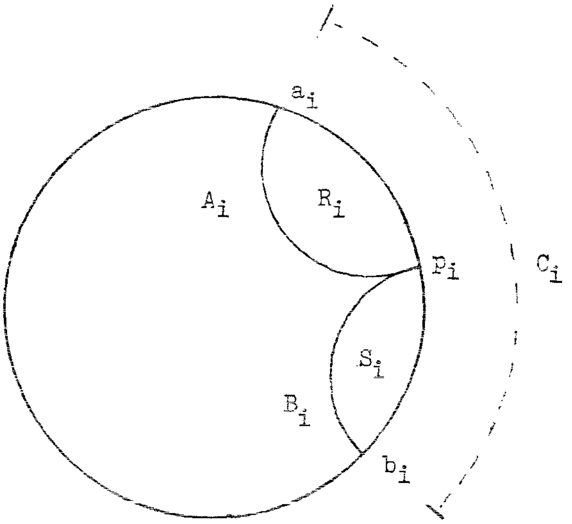


FIGURE 1

Proof Let A_i be the hyperbolic line in D (that is, the arc of a circle orthogonal to U) from a_i to p_i; then h_i carries A_i into the hyperbolic line B_i from b_i to p_i. Let R_i and S_i be the region bounded by C_i and, respectively, A_i and B_i. Then $h_i(D-R_i) \subset S_i$ and $h_i^{-1}(D_i - S_i) \subset R_i$. It follows that, if $i \neq j$, and $m \neq 0$, $h_i^m(D-(R_i \cup S_i)) \subset D-(R_j \cup S_j)$. Let $w = h_{i_t}^{m_t} \ldots h_{i_1}^{m_1}$, $t \geqslant 1$, $m_{i_h} \neq 0$, $i_h \neq i_{h+1}$ be an arbitrary non-trivial reduced word. By induction, if $T = D - \bigcup_{i=1}^{n} (R_i \cup S_i)$, then $wT \subseteq R_{i_t} \cup S_{i_t}$. ∎

1.2 THEOREM (Sanov, Brenner) *Let* $h_1 = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}$, $h_2 = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$. *If* $m \geqslant 2$, *then h_1 and h_2 generate a free group.*

Proof Note that $h_1(m/2) = -m/2$, $h_2(-2/m) = 2/m$, ∞ is the only fixed point of h_1 and 0 is the only fixed point of h_2. The action of h_1 and h_2 on the disk is therefore given by the following simple picture (Figure 2) and the result follows from 1.1. ∎

If $|m| < 2$, the picture changes (Figure 3) for the arms cross one another. Let p and q be the points where the arms cross. By symmetry $h_1^{-1}(p) = h_2(p)$

$= q$, where p and q are the points at which the arcs intersect. Then p is a fixed point of $h_1 h_2$. Let α denote the angle shown between the two arms of h_1 and h_2 at p. It is easy to see that $h_1 h_2$ rotates the southeast arm by 2α. Therefore, if α is a rational multiple of π, which will occur at a dense set of values for m, some $(h_1 h_2)^n$ will be the identity in $PSL(2, R)$, hence $(h_1 h_2)^{2n} = I$. Thus we have:

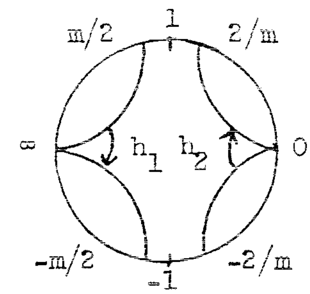


FIGURE 2

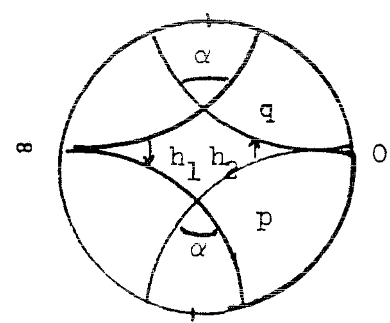


FIGURE 3

1.3. THEOREM (Chang, Jennings, Ree) *The values of m for which h_1 and h_2 generate a group containing torsion are dense in* $[-2, 2]$.

The previous arguments inspire immediate generalizations, such as

1.4 PROPOSITION *If $m > 2$, then, for any $n \geqslant 2$ there is a rank n free subgroup of $SL(2, R)$ containing $G(m)$ as a free factor.*

Proof In the arc $(2/m, m/2)$ introduce points $p_3, \ldots, p_n$ and parabolic transformations $h_3, \ldots, h_n$, with p_i the fixed point of h_i, chosen so that for $C_1 = (m/2, -m/2)$, $C_2 = (-2/m, 2/m)$, arcs $C_j, \ldots, C_n$ can be found as in 1.1 with $C_i \cap C_j = \emptyset$ for $i \neq j$. ∎

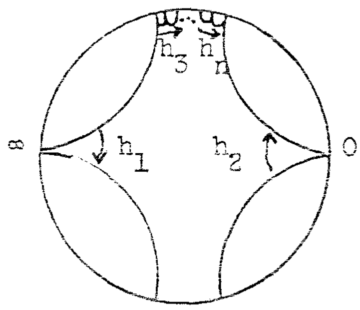


FIGURE 4

2. THE GROUPS $G(\alpha, \beta, \gamma)$

Here is another class of free subgroups. Let $\theta_0, \theta_1, \theta_2$ be three positive angles whose sum is $\pi/2$. Draw three diameters of the circle forming angles $2\theta_0, 2\theta_1, 2\theta_2$ as shown in Figure 5.

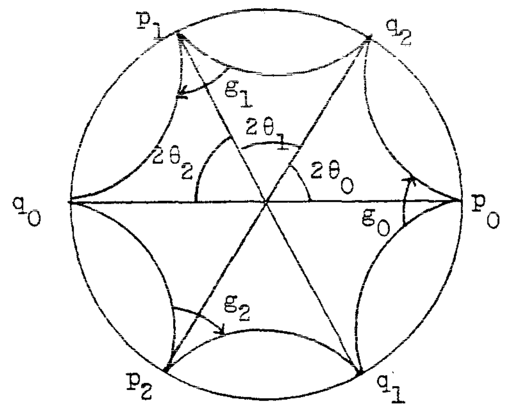


FIGURE 5

Label the endpoints of the ith diameter p_i and q_i, so that the p_i's and q_i's alternate around the circle. Let g_i be the parabolic transformation which fixes p_i and carries q_{i+1} to q_{i+2} (subscripts taken mod 3). Then, according to 1.1, the group generated by g_0, g_1 and g_2 is free. Furthermore

2.1 PROPOSITION *The subgroup of $SL(2, R)$ generated by g_0, g_1 and g_2 is conjugate to $G(\alpha, \beta, \gamma)$ with $\alpha = \cot\theta_0 \cot\theta_2$, $\beta = \cot\theta_2 \cot\theta_1$, $\gamma = \cot\theta_0 \cot\theta_1$.*

Proof With no loss of generality we may assume $p_0 = 0$, $p_1 = \tan(\theta_0 + \theta_1) = \cot\theta_2$ and $p_2 = -\tan(\theta_1 + \theta_2) = -\cot\theta_1$.

Let

$$T = (p_1p_2(p_2-p_1))^{\frac{1}{2}}\begin{pmatrix} p_1 & -p_1p_2 \\ p_2 & -p_1p_2 \end{pmatrix}$$

Then $T(p_0) = 1$, $T(p_1) = \infty$ and $T(p_2) = 0$. Let $g_0 = \begin{pmatrix} 1 & 0 \\ \sigma_0 & 1 \end{pmatrix}$ where

$$\sigma_0 = \frac{\tan\theta_0 + \tan\theta_2}{\tan\theta_0 \tan\theta_2} = \cot\theta_0 + \cot\theta_2.$$

Then

$$Tg_0T^{-1} = \begin{pmatrix} 1+\dfrac{\sigma_0 p_1 p_2}{p_2-p_1} & \dfrac{-\sigma_0 p_1 p_2}{p_2-p_1} \\ \dfrac{\sigma_0 p_1 p_2}{p_2-p_1} & 1-\dfrac{\sigma_0 p_1 p_2}{p_2-p_1} \end{pmatrix} = \begin{pmatrix} 1+\alpha & -\alpha \\ \alpha & 1-\alpha \end{pmatrix}$$

where $\alpha = \cot\theta_0 \cot\theta_2$.

Similarly let $U_i = (1+p_i^2)^{\frac{1}{2}}\begin{pmatrix} 1 & -p_i \\ p_i & 1 \end{pmatrix}$, which rotates the disk around its center, bringing p_i to 0, let $\sigma_1 = \cot\theta_1 + \cot\theta_2$, $\sigma_2 = \cot\theta_0 + \cot\theta_2$, and let $g_i = U_i^{-1}\begin{pmatrix} 1 & 0 \\ \sigma_i & 1 \end{pmatrix}U_i$. Then

$$Tg_1T^{-1} = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}$$

where $\beta = \cot\theta_2 \cot\theta_1$. Similarly

$$Tg_2T^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

where $\gamma = \cot\theta_0 \cot\theta_1$. ∎

Now suppose that $\alpha' \geqslant \alpha$, $\beta' \geqslant \beta$, $\gamma' \geqslant \gamma$. Then T^{-1} conjugates $G(\alpha', \beta', \gamma')$ to a configuration as shown in Figure 6. The fixed points p_0, p_1, p_2 are the same, but

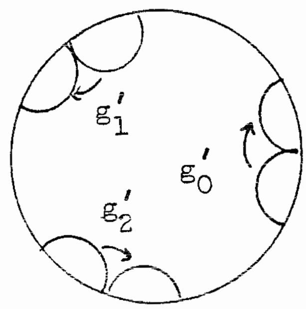


FIGURE 6

the arms defining the action of g'_0, g'_1, g'_2 have shrunk. The argument of 1.1 shows $G(\alpha', \beta', \gamma')$ will still be free, so α, β, γ should be regarded as a lower bound for which we know the group is free. In particular:

2.2 THEOREM *If* $1/\alpha+1/\beta+1/\gamma \leqslant 1$ *then* $G(\alpha, \beta, \gamma)$ *is free.*

Proof From the preceding remarks we may assume equality holds. Let $\theta_0, \theta_1, \theta_2$ be defined by $0 < \theta_0, \theta_1, \theta_2 < \pi/2$, $\cot\theta_0 = \sqrt{\alpha\gamma/\beta}$, $\cot\theta_2 = \sqrt{\alpha\beta/\gamma}$. Since $\alpha > 1$, $\cot\theta_0 > 1/\cot\theta_2 = \tan\theta_2$ so $0 < \theta_0+\theta_2 < \pi/2$; let $\theta_1 = \pi/2-\theta_0-\theta_2$. Then

$$\cot\theta_1 = \tan(\theta_0+\theta_2) = \frac{(\cot\theta_2+\cot\theta_0)}{(\cot\theta_2\cot\theta_0-1)} = \left(\frac{\gamma+\beta}{\alpha-1}\right)\sqrt{\frac{\alpha}{\gamma\beta}},$$

so

$$\cot\theta_1\cot\theta_0 = \frac{(\gamma+\beta)\alpha}{(\alpha-1)\beta} = \gamma, \quad \cot\theta_1\cot\theta_2 = \beta, \quad \cot\theta_0\cot\theta_2 = \alpha.$$

Thus the group constructed above for $\theta_0, \theta_1, \theta_2$ is free and conjugate to $G(\alpha, \beta, \gamma)$. ∎

On the other hand:

2.3 THEOREM *The values of* α, β, γ *for which* $G(\alpha, \beta, \gamma)$ *is not even torsion free are dense in the region* $1/\alpha+1/\beta+1/\gamma > 1$.

Proof Suppose $1/\alpha+1/\beta+1/\gamma > 1$. Let the matrices h_0, h_1, h_2 be the generators of $G(\alpha, \beta, \gamma)$ defined in the introduction. Then $\mathrm{Tr}\,(h_1h_0h_2) = 2+\alpha\beta\gamma(1-1/\alpha-1/\beta-1/\gamma) < 2$. Therefore $h_1h_0h_2$ is an elliptic element, and a hyperbolic rotation through θ, where $2\cos\theta = \mathrm{Tr}\,(h_1h_0h_2)$. For certain arbitrarily close values of α, β, γ, the angle θ will be a rational multiple of π and $h_1h_0h_2$ will be of finite order. ∎

3. CONJUGATE SUBGROUPS

When are two subgroups of the form $G(\alpha, \beta, \gamma)$ conjugate in $SL(2, R)$? The matrix $S_0(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & 0 \\ 1 & \beta-1 \end{pmatrix}$ conjugates

$$h_1 \quad \text{to} \quad \begin{pmatrix} 1+\beta/(\beta-1) & -\beta/(\beta-1) \\ \beta/(\beta-1) & 1-\beta/(\beta-1) \end{pmatrix}$$

$$h_1h_0h_1^{-1} \quad \text{to} \quad \begin{pmatrix} 1 & -\alpha(\beta-1) \\ 0 & 1 \end{pmatrix}$$

and

$$h_2 \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ (\beta-1)\gamma & 1 \end{pmatrix},$$

hence $G(\alpha, \beta, \gamma)$ to $G(\beta/\beta-1, \alpha(\beta-1), \gamma(\beta-1))$. Similarly

$$S_1(\alpha, \beta, \gamma) = \begin{pmatrix} \gamma & -1 \\ \gamma-1 & 0 \end{pmatrix}$$

conjugates

$$h_2 \quad \text{to} \quad \begin{pmatrix} 1 & -\dfrac{\gamma}{\gamma-1} \\ 0 & 1 \end{pmatrix}$$

$$h_2 h_1 h_2^{-1} \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ \beta(\gamma-1)1 & 1 \end{pmatrix}$$

and

$$h_0 \quad \text{to} \quad \begin{pmatrix} 1+\alpha(\gamma-1) & -\alpha(\gamma-1) \\ \alpha(\gamma-1) & 1-\alpha(\gamma-1) \end{pmatrix},$$

hence $G(\alpha, \beta, \gamma)$ to $G(\alpha(\gamma-1), \gamma/\gamma-1, \beta(\gamma-1))$.

In this section we prove the converse. Let τ_0 and τ_1 be the automorphisms of the region $1/\alpha+1/\beta+1/\gamma \leqslant 1$, $\alpha, \beta, \gamma > 0$ given by

$$\tau_0(\alpha, \beta, \gamma) = (\beta/(\beta-1), \alpha(\beta-1), \gamma(\beta-1)).$$

$$\tau_1(\alpha, \beta, \gamma) = (\alpha(\gamma-1), \gamma/(\gamma-1), \beta(\gamma-1)).$$

Let $\mathcal{T}$ denote the group of automorphisms of the region generated by τ_0 and τ_1.

3.1. Theorem *In the region* $1/\alpha+1/\beta+1/\gamma \leqslant 1$, $\alpha, \beta, \gamma > 0$, $G(\alpha, \beta, \gamma)$ *is conjugate to* $G(\alpha', \beta', \gamma')$ *if and only if* (α, β, γ) *and* $(\alpha', \beta', \gamma')$ *are equivalent under the action of* $\mathcal{T}$.

Suppose $1/\alpha+1/\beta+1/\gamma \leqslant 1$ and consider the conjugate of $G(\alpha, \beta, \gamma)$ constructed above, with generators g_i, $i = 0, 1, 2$. Let Δ be the closed

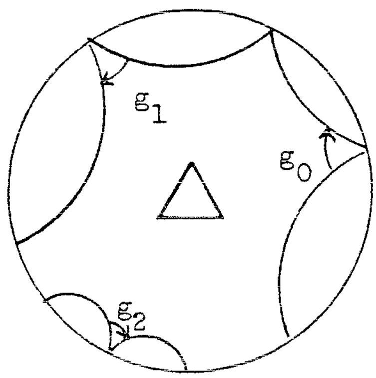


FIGURE 7

(in $\mathring{H}$) region bounded by the arms of g_i in the construction. We have shown that, for w a word in the g_i, $w(\mathring{\Delta}) \cap \mathring{\Delta} = \emptyset$ unless $w = 1$. In fact

3.2 LEMMA *Δ is a fundamental domain for $G(\alpha, \beta, \gamma)$, that is, $H = \bigcup_w w(\Delta)$.*

Proof First consider the case $1/\alpha + 1/\beta + 1/\gamma < 1$, when arms for the g_i can be chosen so that they are disjoint except at the points p_i in $R^+ = R \cup \{\infty\}$. Notice that g_i will map any circle σ_i in H tangent to the real axis at p_i onto itself; this is most easily seen by conjugating p_i to ∞, in which case g_i is the addition of a real constant and σ_i is a line parallel to the real axis in the upper half plane. Such circles are called horocycles [T]. Choose σ_i, $i = 0, 1, 2$, so small that they are disjoint and intersect Δ in a single arc. Then some g_i^n will cover any point in the disk which σ_i bound by a point in Δ. Thus the lemma is true for points x in Δ^+, the union of Δ and the disks bounded by the σ_i.

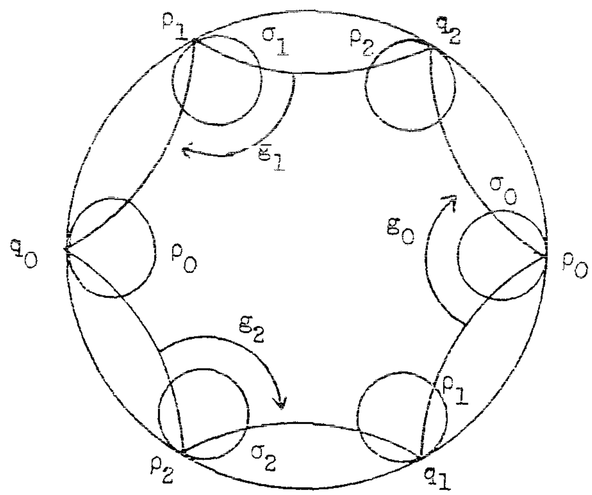


FIGURE 8

The boundary of Δ^+ in $\mathring{H}$ consists of six arcs and, by considering again the special case when $p_i = \infty$, it is easy to see that for some $\varepsilon > 0$ distinct arcs and distinct σ_i will be a distance greater than ε apart. Let $r: [0, l] \to \mathring{H}$ be a unit speed geodesic from a point p in Δ to x. Suppose for some t in $[0, l]$ and some word w in the g_i, $wr(t)$ is in Δ^+. Let t_0 be the value of t at which wr (t) leaves Δ^+. If wr (t_0) is in $\Delta \cap \Delta^+$ then wr (t_0) lies on an arm of some g_i, so $g_i^{\pm 1}$ wr (t) will be entering Δ^+ at t_0. It is easy to see that this geodesic must leave Δ^+ (if it does leave) in a different arc of $\partial\Delta^+$, so the lemma will be true for x in $r[t_0, t_0 + \varepsilon]$. If wr (t_0) is in some σ_i then some g_i^n wr (t_0) is in

$\sigma_i \cap \Delta$ and this geodesic will leave Δ^+ either in some arc of $\Delta \cap \Delta^+$ or after passing through a $\sigma_j \neq \sigma_i$. Hence in this case too the lemma is true for x in $r[t_0, t_0+\varepsilon]$. By repeated application of this argument the lemma is true for all of $r[0, l]$.

In case $1/\alpha+1/\beta+1/\gamma = 1$ the arcs of Δ^+ will no longer be a positive distance apart, for the arms of the g_i will intersect at points other than ρ_i in R^+, e.g. at the point $q_0 = \infty$. Let ρ_0 be a small circle tangent to R^+ at ∞ and let $\rho_1 = g_2(\rho_0)$, $\rho_2 = g_0 g_2(\rho_0)$. It is obvious from the geometry that $g_1 g_0 g_2(q_0) = q_0$ and, since $\operatorname{tr}(g_1 g_0 g_2) = 2$ (see 2.3), q_0 is the only fixed point of $g_1 g_0 g_2$. Therefore, as in the case of σ_i above, $g_1 g_0 g_2(\rho_0) = \rho_0$. Expand Δ^+ to include disks bounded by the ρ_i. The proof then follows as in the previous case. ∎

Remark The lemma was proven by showing that $\bigcup_{g\in G} \Delta$ is "geodesically complete", i.e. any geodesic can be extended indefinitely. This is an *ad hoc* version of the more general "closed horocycle theorem" [T].

Suppose $G \subset SL(2, R)$ is a discrete subgroup; that is, there is an $\varepsilon > 0$ such that the identity matrix is the only element of G whose entries differ from that of the identity by less than ε. Suppose, furthermore, that $g(x) \neq x$ for any $x \in \mathring{H}$ and any non-trivial g in G. Then any $x \in \mathring{H}$ has a neighborhood U such that $g(U) \cap U = \emptyset$ whenever $g \in G$ is non-trivial, so the quotient space H/G is a manifold, with a metric induced from that of H.

3.3 PROPOSITION *If* $1/\alpha+1/\beta+1/\gamma \leqslant 1$ *then* $Q = \mathring{H}/G(\alpha, \beta, \gamma)$ *is homeomorphic to a sphere with* 4 *punctures. Equality holds if and only if the area of* Q *is finite; in this case the area is* 4π.

Proof By the previous lemma, Q is homeomorphic to the quotient space of Δ by the equivalence relation induced by $G(\alpha, \beta, \gamma)$. On the other hand, the proof that $G(\alpha, \beta, \gamma)$ is free shows that the only points of Δ identified by $G(\alpha, \beta, \gamma)$ are points identified by g_i in the arms of g_i. This quotient space is clearly the 4-punctured sphere one puncture for each p_i and a fourth the common image of the ends of all the arms. The area is that of Δ, which is finite if and only if $1/\alpha+1/\beta+1/\gamma = 1$. In this case Δ can be divided into 4 triangles each with geodesic sides and trivial angles, since all arcs are orthogonal to U (Figure 9). By the Gauss–Bonnett theorem in the hyperbolic metric, the area of Q is 4π. ∎

The map $\mathring{H} \to Q$ defined above is a covering space. Choose a base point q in Q and let $\pi_1(Q)$ be the fundamental group of Q at q. Then $\pi_1(Q)$ is free of rank 3, and the natural isomorphism of the group $G(\alpha, \beta, \gamma)$ in $PSL(2, R)$ onto $\pi_1(Q)$ carries the generator h_i of $G(\alpha, \beta, \gamma)$, for $i = 0, 1, 2$, to the element l_i of $\pi_1(Q)$ represented by a loop at q running from q to the puncture p_i, once around p_i, and back to q.

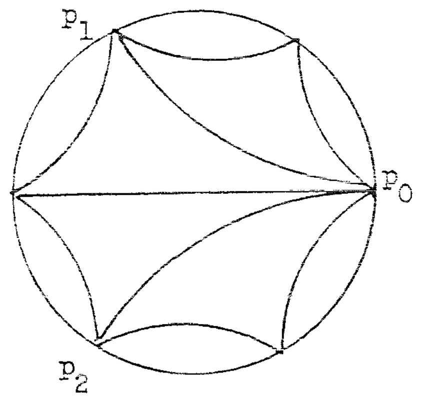


FIGURE 9

Proof of 3.1 Let $T \in SL(2, R)$ and suppose $TG(\alpha, \beta, \gamma) T^{-1} = G(\alpha', \beta', \gamma')$. Since $1/\alpha + 1/\beta + 1/\gamma \leqslant 1$, no $g \in G(\alpha, \beta, \gamma)$ has a fixed point in $\mathring{H}$, so no $g' \in G(\alpha', \beta', \gamma')$ has a fixed point in $\mathring{H}$. Therefore $1/\alpha' + 1/\beta' + 1/\gamma' \leqslant 1$ (see 2.3). Let Q' denote the quotient $\mathring{H}/G(\alpha', \beta', \gamma')$, also a 4-punctured sphere.

The transformation $T: \mathring{H} \to \mathring{H}$ induces an isometry $\overline{T}: Q \to Q'$. Thus by 3.3, $1/\alpha + 1/\beta + 1/\gamma = 1$ if and only if $1/\alpha' + 1/\beta' + 1/\gamma' = 1$.

Case 1 $1/\alpha + 1/\beta + 1/\gamma < 1$.

Regard Q and Q' as the complement of four points in S^2. One of the four punctures $p_3(p_3')$ is distinguished geometrically by having no neighborhood in S^2 of finite area in $Q(Q')$. Thus $\overline{T}: Q \to Q'$ extends to a homeomorphism $\overline{T}: S^2 \to S^2$ such that $\overline{T}\{p_0, p_1, p_2\} = \{p_0', p_1', p_2'\}$, $\overline{T}(p_3) = p_3'$. There is an evident homeomorphism $f: S^2 \to S^2$ such that $f(p_3) = p_3'$ and $f(l_i) = l_i'$, $i = 0, 1, 2$. Then $f^{-1}\overline{T}: S^2 \to S^2$ fixes p_3. It is well-known that any homeomorphism $f^{-1}\overline{T}: (S^2, p_3) \to (S^2, p_3)$ is isotopic to the identity [Bi]. The image of $\{p_0, p_1, p_2\} \times I$ under the isotopy in $S^2 \times I$ is a braid of 3-strands, so the automorphism $(f^{-1}\overline{T})_{\#}: \pi_1(Q) \to \pi_1(Q)$ is induced by the braid group on 3 strands B_3. This group is generated by the automorphisms σ_i, $i = 0, 1$,

$$\sigma_i(l_i) = l_{i+1}, \qquad \sigma_i(l_{i+1}) = l_{i+1} l_i l_{i+1}^{-1}$$
$$\sigma_i(l_j) = l_j \quad \text{for} \quad j \neq i$$

with the relation $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$ [Bi], [MKS, p. 173]. With no loss of generality assume $(f^{-1}T)_{\#}$ is the automorphism σ_i, $i = 0, 1$. Compose with $f_{\#}$, then conjugate by the matrix $S_i(\alpha', \beta', \gamma')$ defined at the beginning of

Section 3. Then S_iT carries the fixed points $0, 1, \infty$ of $G(\alpha, \beta, \gamma)$ to themselves so $S_iT = \pm I$. Then T is in $\mathscr{T}$, proving the proposition in this case.

Case 2 $1/\alpha+1/\beta+1/\gamma = 1$.

It is no longer possible to distinguish the puncture p_3 from the others geometrically so it is no longer possible to guarantee that $\bar{T}(p_3) = p_3'$. Suppose, for example, that $\bar{T}(p_0) = p_3'$; we will show that there is an $S \in SL(2, R)$ which conjugates $G(\alpha, \beta, \gamma)$ to *itself* such that $\bar{S}(p_3) = p_0$, $\bar{S}(p_2) = p_1$, $\bar{S}(p_1) = p_2$. Then $\bar{T}\bar{S}(p_3) = p_3$ and the proof proceeds as before.

Since h_i represents l_i in $G(\alpha, \beta, \gamma)$, $(h_1h_0h_2)^{-1}$ represents a curve going once around p_3. Since $1/\alpha+1/\beta+1/\gamma = 1$,

$$h_1h_0h_2 = \begin{pmatrix} 1+\alpha & \alpha\beta/\gamma \\ -\gamma\alpha/\beta & 1-\alpha \end{pmatrix}$$

Let

$$S = \begin{pmatrix} 0 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & 0 \end{pmatrix}$$

Then $S(h_1h_0h_2)^{-1}S^{-1} = h_0$, $Sh_2S^{-1} = h_2$ and $Sh_1S^{-1} = h_2$ as required.

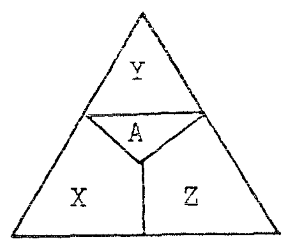
The proof is similar for $T^{-1}(p_3') = p_1$ or p_2.

PROPOSITION 3.2 $\mathscr{T} \cong PSL(2, Z)$.

Proof It follows from the proof of 3.1 that $\mathscr{T}$ is a factor group of the braid group $\langle \tau_0, \tau_1 : \tau_1\tau_0\tau_1 = \tau_0\tau_1\tau_0 \rangle$. It is easy to check that $\tau_0\tau_1(\alpha, \beta, \gamma) = (\gamma, \alpha, \beta)$ so $(\tau_0\tau_1)^3 = 1$. Thus $\mathscr{T}$ is a factor group of $\langle \tau_0, \tau_1 : (\tau_0\tau_1)^3 = 1 = (\tau_1\tau_0\tau_1)^2 \rangle \cong Z_2 * Z_3 \cong PSL(2, Z)$.

To produce an isomorphism it suffices to show that the generators $\tau_0\tau_1$ and τ_1 of $\mathscr{T}$ satisfy no relation other than $(\tau_0\tau_1)^3 = (\tau_1\tau_0\tau_1)^2 = 1$. Let $x = 1/\alpha$, $y = 1/\beta$, $z = 1/\gamma$, σ_0 be the induced action of $\tau_0\tau_1$ on (x, y, z) and σ_1 the action of $\tau_1\tau_0\tau_1$, so

$$\sigma_0(x, y, z) = (z, x, y), \qquad \sigma_1(x, y, z) = \left(\frac{zy}{1-y}, 1-y, \frac{xy}{1-y}\right).$$



Consider the open simplex $x, y, z > 0$, $x+y+z = 1$. Let $X = \{x > y, z\}$, $Y = \{y > \frac{1}{2}\}$, $Z = \{z > y, x\}$, $A = \{\frac{1}{2} > y > x, z\}$, disjoint regions in the simplex. Let w be a word in σ_0 and σ_1, reduced using the relations $\sigma_0^3 = \sigma_1^2 = 1$ so that all exponents are positive and as small as possible. Then $w(A) \subset Y$ if w begins with σ_1, $w(A) \subset Z$ if w begins with $\sigma_0\sigma_1$ and $w(A) \subset X$ if w begins with σ_0^2. The proof is an easy induction, as in 1.1, using the inclusions $\sigma_1(X \cup Z \cup A) \subset Y$, $\sigma_0\sigma_1(X \cup A \cup Z) \subset Z$ and $\sigma_0^2(Y \cup A) \subset X$. In particular $w(A) \cap A = \emptyset$ unless w is trivial, so $\mathscr{T}$ has no further relations.

Problem There is apparently a topological conjugacy of $\mathscr{T}$ acting on the open simplex to $PSL(2, Z)$ acting on $\mathring{H}$. Construct an explicit homeomorphism from the simplex to $\mathring{H}$ realizing this conjugacy.

4. THE INCLUSION PROBLEM

Bachmuth and Mochizuki ask whether $G(\alpha, \beta, \gamma)$ is always contained in a rank two free subgroup of $SL(2, R)$. In this section we prove:

4.1 THEOREM *For α, β, γ in the region $1/\alpha+1/\beta+1/\gamma = 1$, $2 \leqslant \beta \leqslant \alpha, \gamma$, $G(\alpha, \beta, \gamma)$ is contained in a free rank two discrete subgroup of $SL(2, R)$ if and only if $(\alpha, \beta, \gamma) = (4, 2, 4)$.*

One direction is easy, following from the more general

4.2 PROPOSITION *If $\beta = 2$ and $\alpha = \gamma$, then $G(\alpha, \beta, \gamma)$ is conjugate in $SL(2, R)$ to a subgroup of $G(m)$, $m^2 = \alpha = \gamma$.*

Proof Let $U = \begin{pmatrix} 1/\sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Then

$$UaU^{-1} = h_2$$
$$Ub^2U^{-1} = h_1^{-1}$$
$$Uvab^{-1}U^{-1} = h_0.$$

Proof of 4.1 Suppose $G = G(\alpha, \beta, \gamma)$ is contained in a rank two free discrete subgroup $\hat{G}$ of $PSL(2, R)$. Since $\mathscr{T}$ is free and discrete it contains no elliptic elements. Therefore $\hat{G}$ acts freely on $\mathring{H}$ and $\mathring{H}/\hat{G}$ is a surface M. Since $G \subset \hat{G}$, $Q = \mathring{H}/G$ is a covering space of M. By 3.3, Q has finite area, so M does also. The index of G in $\hat{G}$ is the ratio of the areas of Q and M, and by the Schreier formula [MKS, 2.10] this index must be two. Since $\pi_1(M) \cong \hat{G} \cong Z * Z$, M is either a 3-punctured sphere or a punctured torus of area 2π.

Actually M is a 3-punctured sphere, for no punctured sphere can be a finite cover of a punctured torus. This is best seen by observing that a meridian and longitude of the torus intersect once, hence in an n-fold cover their lifts will intersect n times with the same orientation. But the orientations of points of intersection of loops in S^2 cancel, since S^2 is simply-connected.

Denote the punctures in M by p_1, p_2 and q and choose disjoint imbedded geodesics $\alpha_i\colon R \to M$, $i = 1, 2$, such that $\lim_{t\to\infty} \alpha_i(t) = p_i$, $\lim_{t\to-\infty} \alpha_i(t) = q$. Let $\hat{\Delta}$ denote the closure of a lift of $M-(\alpha_1(R) \cup \alpha_2(R))$ in $\mathring{H}$. Then $\hat{\Delta}$ is a fundamental domain for $\hat{G}$ whose boundary consists of four hyperbolic lines. Since the area of $\hat{\Delta}$ is finite, adjacent hyperbolic lines have a common endpoint on U. Each generator f_i of $\hat{G}$ identifies an adjacent pair of hyperbolic lines whose common endpoint $\hat{p}_i$ in U is fixed by f_i. If f_i fixed a second point on U then the hyperbolic line between the fixed points would project to a geodesic circle running once around p_i. The triangle near p_i, cut out by this circle and α_i, would contradict the Gauss–Bonnet theorem (its angles sum to π). Hence $\hat{p}_i$ is the only fixed point of f_i.

Conjugate $\hat{G}$ by an element of $PSL(2, R)$ so that the fixed points of f_1 and f_2 are at 0 and ∞. Then $f_1 = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$. Using f_i^{-1}, if necessary, we may assume $a_1, a_2 > 0$. Conjugation by

$$\begin{pmatrix} \sqrt[4]{a_2/a_1} & 0 \\ 0 & \sqrt[4]{a_2/a_1} \end{pmatrix}$$

will assure that $a_1 = a_2$. It was shown in 1.2 that if $a_1 = a_2 < 2$ then $\hat{G}$ fixes a point in $\mathring{H}$ and thus will either not be free or not be discrete. If $a_1 = a_2 > 2$ then M would have infinite area (the proof is analogous to 3.3). Thus G is conjugate in $PSL(2, R)$ to an index two subgroup of the Sanov subgroup $G(2)$.

In general, suppose N is an index two subgroup of the free group F generated by x and y. Then N properly contains the kernel N_1 of the obvious map from F to $Z_2 \oplus Z_2$. Since N_1 is generated by x^2, y^2 and $(xy)^2$, N must be obtained by adjoining one of x, y, or xy to the given generators of N_1. Thus the only index two subgroups of $G(2)$ are generated by $\{f_1^2, f_2, (f_1f_2)^2\}$, $\{f_1, f_2^2, (f_1f_2)^2\}$ and $\{f_1^2\, f_2^2, f_1f_2\}$. It is easy to check that each is conjugate to $G(4, 2, 4)$.

Now the region described in the theorem coincides with the closure of the region A described in the proof of 3.2. From that proof it follows that no other $G(\alpha, \beta, \gamma)$ from that region is conjugate to $G(4, 2, 4)$, proving the theorem.

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