PROXIMITY IN THE CURVE COMPLEX: BOUNDARY REDUCTION AND BICOMPRESSIBLE SURFACES

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ABSTRACT. Suppose N is a compressible boundary component of a compact irreducible orientable 3-manifold M and $(Q,\partial Q) \subset (M,\partial M)$ is an orientable properly embedded essential surface in M in which some essential component is incident to N and no component is a disk. Let $\mathcal{V}$ and $\mathcal{Q}$ denote respectively the sets of vertices in the curve complex for N represented by boundaries of compressing disks and by boundary components of Q.

Theorem: Suppose Q is essential in M, then $d(\mathcal{V},\mathcal{Q}) \leq 1-\chi(Q)$.

Hartshorn showed ([Ha]) that an incompressible surface in a closed 3-manifold puts a limit on the distance of any Heegaard splitting. An augmented version of the theorem above leads to a version of Hartshorn's result for merely compact 3-manifolds.

In a similar spirit, here is the main result:

Theorem: Suppose a properly embedded connected surface Q is incident to N. Suppose further that Q is separating and compresses on both its sides, but not by way of disjoint disks. Then either

- $d(\mathcal{V},\mathcal{Q}) \leq 1-\chi(Q)$ or
- Q is obtained from two nested connected incompressible boundary-parallel surfaces by a vertical tubing.

Forthcoming work with M. Tomova ([STo]) will show how an augmented version of this theorem leads to the same conclusion as in Hartshorn's theorem, not from an essential surface but from an alternate Heegaard surface. That is, if Q is a Heegaard splitting of a compact M then no other Heegaard splitting has distance greater than twice the genus of Q.

1. INTRODUCTION

Suppose N is a compressible boundary component of an orientable irreducible 3-manifold M and $(Q,\partial Q) \subset (M,\partial M)$ is an essential orientable

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surface in M in which an essential component is incident to N and no component of Q is a disk. Let $\mathcal{V}, \mathcal{Q}$ denote sets of vertices in the curve complex for N represented respectively by boundaries of compressing disks and by boundary components of Q. We will show:

- The distance $d(\mathcal{V}, \mathcal{Q})$ in the curve complex of N is no greater than $1 - \chi(Q)$. Furthermore, if no component of Q is an annulus ∂-parallel into N, then for each component q of $Q \cap N$, $d(q, \mathcal{V}) \leq 1 - \chi(Q)$.

A direct consequence is this generalization of a theorem of Hartshorn [Ha]:

- If S is a Heegaard splitting surface for a compact orientable manifold M and $(Q, \partial Q) \subset (M, \partial M)$ is a properly embedded incompressible surface, then $d(S) \leq 2 - \chi(Q)$.

Both results are unsurprising, and perhaps well-known (see eg [BS] for discussion of this in the broader setting of knots in bridge position with respect to a Heegaard surface).

It would be of interest to be able to prove the second result (Hartshorn's theorem) for Q a Heegaard surface, rather than an incompressible surface. Of course this is hopeless in general: a second copy of P could be used for Q and that would in general provide no information about the distance of the splitting P at all. But suppose it is stipulated that Q is not isotopic to P. One possibility is that Q is weakly reducible. In that case (cf [CG]) it is either the stabilization of a lower genus Heegaard splitting (which we revert to) or it gives rise to a lower genus incompressible surface and this allows the direct application of Hartshorn's theorem. So in trying to extend Hartshorn's theorem to Q a Heegaard surface, it suffices to consider the case in which Q is strongly irreducible.

The first step in extending [Ha] to Q a Heegaard surface is carried out here, analogous in the program to the first result above. Specifically, we establish that bicompressible but weakly incompressible surfaces typically do not have boundaries that are distant in the curve complex from curves that compress in M.

- Suppose a properly embedded surface Q is connected, separating and incident to N. Suppose further that Q compresses on both its sides, but not by way of disjoint disks, then either
  - $d(\mathcal{V}, \mathcal{Q}) \leq 1 - \chi(Q)$ or
  - Q is obtained from two nested connected boundary-parallel surfaces by a vertical tubing.

From this result forthcoming work will demonstrate, via a two-parameter argument much as in [RS], that the genus of an alternate Heegaard splitting Q does indeed establish a bound on the distance of P.

Maggy Tomova has provided valuable input to this proof. Beyond sharpening the foundational proposition (Propositions 2.5 and Theorem 5.4) in a very useful way, she provided an improved proof of Theorem 3.1.

2. Preliminaries and first steps

First we recall some definitions and elementary results, most of which are well-known.

Definition 2.1. *A ∂-compressing disk for Q is a disk $D \subset M$ so that ∂D is the end-point union of two arcs, $\alpha = D \cap \partial M$ and $\beta = D \cap Q$, and β is essential in Q.*

Definition 2.2. *A surface $(Q, \partial Q) \subset (M, \partial M)$ is* essential *if it is incompressible and some component is not boundary parallel. An essential surface is* strictly essential *if it has at most one non-annulus component.*

Lemma 2.3. *Suppose $(Q, \partial Q) \subset (M, \partial M)$ is a properly embedded surface and Q' is the result of ∂-compressing Q. Then*

(1) *If Q is incompressible so is Q'.*
(2) *If Q is essential, so is Q'.*

Proof. A description dual to the boundary compression from Q to Q' is this: Q is obtained from Q' by tunneling along an arc γ dual to the ∂-compression disk. (The precise definition of tunneling is given in Section 4.) Certainly any compressing disk for Q' in M is unaffected by this operation near the boundary. Since Q is incompressible, so then is Q'. This proves the first claim.

Suppose now that every component of Q' is boundary parallel and the arc γ dual to the ∂-compression has ends on components Q'_0, Q'_1 of Q' (possibly $Q'_0 = Q'_1$). If γ is disjoint from the subsurfaces P_0 and P_1 of ∂M to which Q'_0 and Q'_1 respectively are parallel then tunneling along γ merely creates a component that is again boundary parallel (to the band-sum of the P_i along γ), contradicting the assumption that not all components of Q are boundary parallel. So suppose γ lies in P_0, say. If both ends of γ lie on Q'_0 (so $Q'_1 = Q'_0$) then the disk $\gamma \times I$ in the product region between Q'_0 and P_0 would be a compressing disk for Q, which contradicts the incompressibility of Q.

Finally, suppose $Q'_1 \neq Q'_0$, so $P_0 \subset P_1$ and γ is an arc in $P_1 - P_0$ connecting ∂P_0 to ∂P_1. P_0 is not a disk, else the arc β in which the ∂-compressing disk intersects Q would not have been essential in Q. So there is an essential simple closed curve $\gamma_0 \subset P_0$ based at the point $\gamma \cap P_0$. Attach a band to

γ_0 along γ to get an arc $\gamma_+ \subset P_1$ with both ends on ∂P_1. Then the disk $E_1 = \gamma_+ \times I$ lying between $P_1 \subset \partial M$ and Q_1' intersects Q in a single arc, parallel in M to γ_+ and lying in the union of the top of the tunnel and Q_0'. This arc divides E_1 into two disks; let E be the one not incident to ∂M. E then has its boundary entirely in Q and since it is essential there, E is a compressing disk for Q, again a contradiction. See Figure 1. From these various contradictions we conclude that at least one of the components of Q' to which the ends of γ is attached is not ∂-parallel, so Q' is essential.

□

Definition 2.4. *Suppose S is a closed orientable surface and $\alpha_0, \ldots, \alpha_n$ is a sequence of essential simple closed curves in S so that for each $1 \leq i \leq n$, α_{i-1} and α_i can be isotoped to be disjoint. Then we say that the sequence is a length n path in the curve complex of S (cf* [He]*).*

The distance $d(\alpha, \beta)$ between a pair α, β of essential simple closed curves in S is the smallest $n \in \mathbb{N}$ so that there is a path in the curve-complex from α to β of length n. Curves are isotopic if and only if they have distance 0.

Two sets of curves $\mathcal{V}, \mathcal{W}$ in S have distance $d(\mathcal{V}, \mathcal{W}) = n$ if n is the smallest distance from a curve in $\mathcal{V}$ to a curve in $\mathcal{W}$.

Proposition 2.5. *Suppose M is an irreducible compact orientable 3-manifold, N is a compressible component of ∂M and $(Q, \partial Q) \subset (M, \partial M)$ is a properly embedded essential surface with $\chi(Q) \leq 1$ and at least one essential component incident to N. Let $\mathcal{V}$ be the set of essential curves in N that bound disks in M and let q be any component of ∂Q.*

- *If Q contains an essential disk incident to N, then $d(\mathcal{V}, q) \leq 1$.*
- *If Q does not contain any disk components, then $d(\mathcal{V}, q) \leq 1 - \chi(Q)$ or Q is strictly essential and q lies in the boundary of a ∂-parallel annulus component of Q.*

Proof. If Q contains an essential disk D incident to N, then $\partial D \in \mathcal{V}$. q may be ∂D or it may be another component of ∂Q but in either case $d(\mathcal{V}, q) \leq 1$.

Suppose Q contains no disks at all and thus $\chi(Q) \leq 0$. Let E be a compressing disk for N in M so that, $|E \cap Q|$ is minimal among all such disks. Circles of intersection between Q and E and arcs of intersection that are inessential in Q can be removed by isotoping E via standard innermost disk and outermost arc arguments, so this choice of E guarantees that E and Q only intersect along arcs that are essential in Q. If in fact they don't intersect at all, then $d(\partial E, q) \leq 1$ for every $q \in \partial Q$ and we are done. Consider, then, an arc β of $Q \cap E$ that is outermost in E, cutting off from E a ∂-compressing disk E_0 for Q that is incident to N. Boundary compressing Q along E_0 gives a new essential (by Lemma 2.3) surface $Q' \subset M$ which can be isotoped so that each component of $\partial Q'$ is disjoint from each component of ∂Q. That

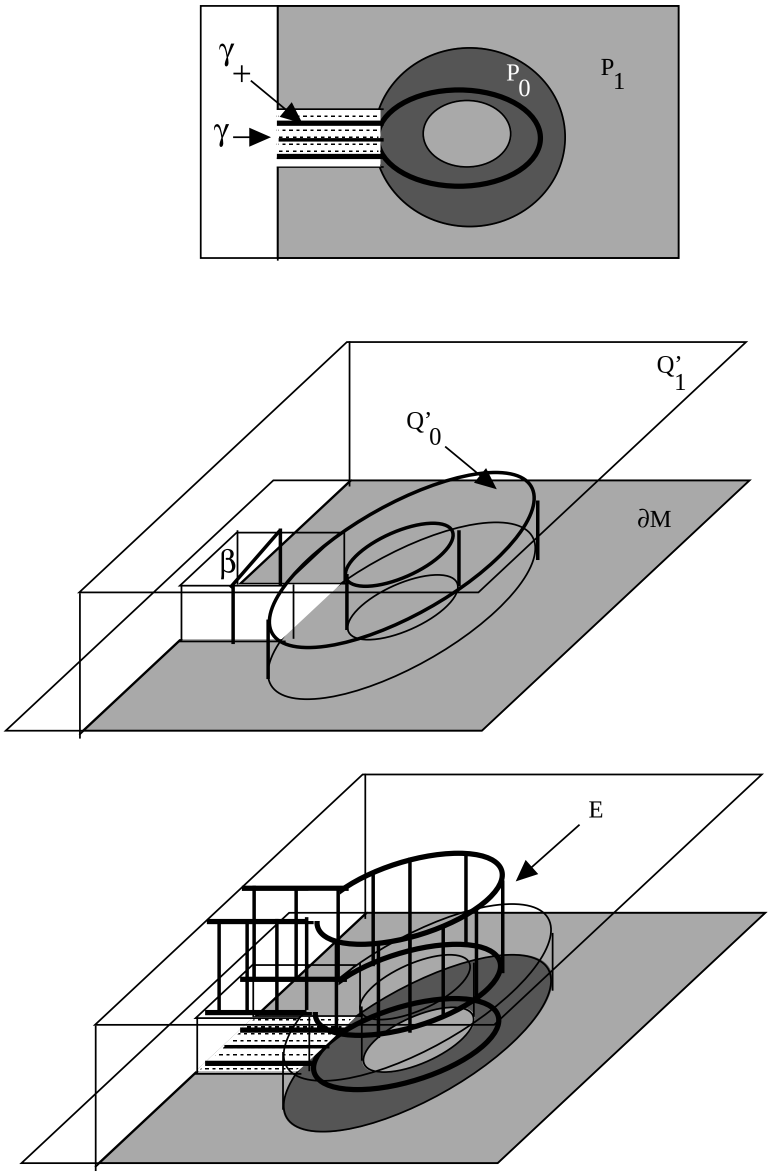

FIGURE 1

is for each component q of ∂Q and each component q' of $\partial Q'$ we have that $d(q,q') \leq 1$.

The proof now is by induction on $1-\chi(Q)$. As Q has no disk components, $1-\chi(Q) \geq 1$. Suppose $1-\chi(Q) = 1$, i.e. all components of Q are annuli, so Q is strictly essential. As we are not making any claims about the curves in $\mathcal{Q}$ coming from ∂-parallel annuli components, we may assume all annuli in Q are essential. Then Q' contains a compressing disk D for N (the result of boundary reducing an essential annulus component of Q along E_0) and ∂D is disjoint from all $q \in \partial Q$. As $\partial D \in \mathcal{V}$, $d(q,\mathcal{V}) \leq 1 = 1-\chi(Q)$ as desired.

Now suppose $1-\chi(Q) > 1$. If Q is not strictly essential, then it contains at least two non-annulus components and, since it is essential, at least one essential component. Thus there is a component Q_0 of Q which is essential and such that $1-\chi(Q_0) < 1-\chi(Q)$. By the induction hypothesis, for each component q_0 of ∂Q_0, $d(q_0,\mathcal{V}) \leq 1-\chi(Q_0)$. Of course also $d(q,q_0) \leq 1$. Combining these inequalities, we obtain the desired result.

Suppose next that Q is strictly essential and again all ∂-parallel annuli have been removed prior to the boundary compression described above. If the boundary compression creates a disk component of Q' then it must be essential and incident to N so $\partial D \in \mathcal{V}$ and for every $q \in \partial Q$, $d(q,\mathcal{V}) \leq d(q,\partial D) \leq 1 \leq 1-\chi(Q)$ and we are done. Suppose then that no component of Q' is a disk and q_1 is any boundary component of an essential component Q_1 of Q'. As $1-\chi(Q_1) \leq 1-\chi(Q') < 1-\chi(Q)$, the induction hypothesis applies and $d(q_1,\mathcal{V}) \leq 1-\chi(Q_1) < 1-\chi(Q)$. Since for every component q of ∂Q, $d(q,q_1) \leq 1$, we have the inequality $d(q,\mathcal{V}) \leq d(q_1,\mathcal{V}) + d(q,q_1) \leq 1-\chi(Q')+1 = 1-\chi(Q)$, as desired. □

In order to prove Hartshorn's theorem on Heegaard splittings it will be helpful to understand what it takes to be an essential surface in a compression body. Recall the definitions (cf [Sc]):

A *compression body* H is a connected 3-manifold obtained from a closed surface $\partial_- H$ by attaching 1-handles to $\partial_- H \times \{1\} \subset \partial_- H \times I$. (It is conventional to consider a handlebody to be a compression body in which $\partial_- H = \emptyset$.) Dually, H is obtained from a connected surface $\partial_+ H$ by attaching 2-handles to $\partial_+ H \times \{1\} \subset \partial_+ H \times I$ and 3-handles to any 2-spheres thereby created. The cores of the 2-handles are called *meridian disks* and a collection of meridian disks is called *complete* if its complement is $\partial_- H \times I$, together perhaps with some 3-balls.

Suppose two compression bodies H_1 and H_2 have $\partial_+ H_1 \simeq \partial_+ H_2$. Then glue H_1 and H_2 together along $\partial_+ H_i = S$. The resulting compact 3-manifold M can be written $M = H_1 \cup_S H_2$ and this structure is called a *Heegaard splitting* of the 3-manifold with boundary M (or, more specifically, of the triple $(M;\partial_- H_1,\partial_- H_2)$). It is easy to show that every compact 3-manifold has a Heegaard splitting.

The following is probably well-known:

Lemma 2.6. *Suppose H is a compression body and $(Q,\partial Q) \subset (H,\partial H)$ is incompressible. If $\partial Q \cap \partial_+ H = \emptyset$, Q is inessential. That is, each component is ∂-parallel.*

Proof. It suffices to consider the case in which Q is connected. To begin with, consider the degenerate case in which $H = \partial_- H \times I$. Suppose there is a counterexample; let Q be a counterexample that maximizes $\chi(Q)$.

Case 1: $H = \partial_- H \times I$ and Q has non-empty boundary.

Q cannot be a disk since $\partial_- H \times I$ is ∂-irreducible, so $\chi(Q) \leq 0$. By hypothesis, $\partial Q \subset \partial_- H \times \{0\}$. Choose $\alpha \subset \partial_- H \times \{0\}$ to be any curve that cannot be isotoped off of ∂Q and let $A = \alpha \times I$ be the corresponding annulus in $\partial_- H \times I$. Consider $Q \cap A$ and minimize by isotopy of A the number of its components. A standard argument shows that there are no inessential circles of intersection and each arc of intersection is essential in Q. Since ∂Q is disjoint from $\partial_- H \times \{1\}$, all arcs of $Q \cap A$ have both ends in $\partial_- H \times \{0\}$. An outermost such arc in A defines a ∂-compression of Q. The resulting surface Q' is still incompressible (for a compressing disk for Q' would persist into Q) and has at most two components, each of higher Euler characteristic and so each ∂-parallel into $\partial_- H$. If there are two components, neither is a disk, else the arc along which ∂-compression was supposedly performed would not have been essential. If there are two components of Q' and they are not nested (that is, each is parallel to the boundary in the complement of the other) it follows that Q was ∂-parallel. If Q' had two nested components, it would follow that Q was compressible, a contradiction. (See the end of the proof of Lemma 2.3 or Figure 1.) Similarly, if Q' is connected then, depending on whether the tunneling arc dual to the ∂-compression lies inside or outside the region of parallelism between Q' and ∂M, Q would either be compressible or itself ∂-parallel.

Case 2: $H = \partial_- H \times I$ and Q is closed.

Let $A = \alpha \times I \subset \partial_- H \times I$ be any incompressible spanning annulus. A simple homology argument shows that Q intersects A. After the standard move eliminating innermost disks, all intersection components are then essential curves in A. Let λ be the curve that is closest in A to $\partial_- H \times \{0\}$. Let Q' be the properly embedded surface (now with boundary) obtained from Q by removing a neighborhood of λ in Q and attaching two copies of the subannulus of A between $\alpha \times \{0\}$ and λ. It's easy to see that Q' is still incompressible and its boundary is still disjoint from $\partial_- H \times \{1\}$, and now Q' has non-empty boundary, so by Case 1, Q' is ∂-parallel. The subsurface of ∂M to which Q' is ∂-parallel can't contain the neighborhood η of $\alpha \times \{0\}$ in ∂M, else the parallelism would identify a compressing disk for Q. It follows that the parallelism is outside of η and so can be extended across η to give a parallelism between Q and a subsurface (hence all) of $\partial_- H \times \{0\}$.

Case 3: The general case.

Let Δ be a complete family of meridian disks for H, so when H is compressed along Δ it becomes a product $\partial_- H \times I$. Since Q is incompressible, a standard innermost disk argument allows Δ to be redefined so that $\Delta \cap Q$ has no simple closed curves of intersection. Since $Q \cap \partial_+ H = \emptyset$ it then follows that $Q \cap \Delta = \emptyset$. Then in fact $Q \subset \partial_- H \times I$ and the result follows from Cases 1) or 2). □

3. HARTSHORN'S THEOREM

Here we give a quick proof of Hartshorn's theorem (actually, an extension to the case in which M is not closed) using Proposition 2.5. Recall that the distance $d(P)$ of a Heegaard splitting ([He]) is the minimum distance in the curve complex of P between a vertex representing a meridian curve on one side of P and a vertex representing a meridian curve on the other side.

Theorem 3.1. *Suppose P is a Heegaard splitting surface for a compact orientable manifold M and $(Q,\partial Q) \subset (M,\partial M)$ is a connected essential surface. Then $d(P) \leq 2 - \chi(Q)$.*

Remark: As long as Q contains no inessential disks or spheres, and at most one essential disk or sphere, Q need not be connected.

Proof. The following are classical facts about Heegaard splittings (cf [Sc]): If Q is a sphere then P is reducible, hence $d(P) = 0$. If Q is a disk then P is ∂-reducible so $d(P) \leq 1$. If neither occurs, then M is irreducible and ∂-irreducible, which is what we henceforth assume. Moreover, once Q is neither a disk nor a sphere then $2 - \chi(Q) \geq 2$ so we may as well assume that $d(P) \geq 2$, ie P is strongly irreducible.

Let A, B be the compression-bodies into which P divides M and let Σ^A, Σ^B be spines of A and B respectively. That is, Σ^A is the union of a graph in A with $\partial_- A$ and Σ^B is the union of a graph in B with $\partial_- B$ so that $M - (\Sigma^A \cup \Sigma^B)$ is homeomorphic to $P \times (-1,1)$. We consider the curves $P \cap Q$ as P sweeps from a neighborhood of Σ^A (i. e. near $P \times \{-1\}$) to a neighborhood of Σ^B (near $P \times \{1\}$). Under this parameterization, let P_t denote $P \times \{t\}$. Consider the possibilities:

Suppose $Q \cap \Sigma^A = \emptyset$. Then Q is an incompressible surface in the compression body $closure(Q - \Sigma^A) \cong B$. By Lemma 2.6, Q would be inessential, so this case does not arise. Similarly we conclude that Q must intersect Σ^B. It follows that when t is near -1, $P_t \cap Q$ contains meridian circles for A; when t is near 1, it contains meridian circles for B. Since P is strongly irreducible, it can never be the case that both occur, so at some generic level neither occurs. (See [Sc] for details, including why we can take such a level to be generic.) Hence there is a generic t_0 so that $P_{t_0} \cap Q$ contains no meridian circles for P.

An innermost inessential circle of intersection in P_{t_0} must be inessential in Q since Q is incompressible. So all such circles of intersection can be removed by an isotopy of Q. After this process, all remaining curves of intersection are essential in P_{t_0}. Since $P_{t_0} \cap Q$ contains no meridian circles for P, no remaining circle of intersection can be inessential in Q either. Hence all components of $P_{t_0} \cap Q$ are essential in both surfaces; in particular no component of $Q - P_{t_0}$ is a disk. At this point, revert to P as notation for P_{t_0}.

If $P \cap Q = \emptyset$ then we are done, just as in the case in which Q is disjoint from a spine. Similarly we are done if the surface $Q_A = Q \cap A$ is inessential (hence ∂-parallel) in A or $Q_B = Q \cap B$ is inessential in B. We conclude that Q_A and Q_B are both essential in A and B respectively, and the positioning of P has guaranteed that no component of either is a disk.

Unless Q_A and Q_B are both strictly essential, the proof follows easily from Proposition 2.5: Suppose, for example, that Q_A is not strictly essential and let $\mathcal{U}, \mathcal{V}$ be the set of curves in P bounding disks in A and B respectively. Let q be a curve in $P \cap Q$ lying on the boundary of an essential component of Q_B. Then Proposition 2.5 says that $d(q, \mathcal{U}) \leq 1 - \chi(Q_A)$ and $d(q, \mathcal{V}) \leq 1 - \chi(Q_B)$ so

$$d(P) = d(\mathcal{U}, \mathcal{V}) \leq d(q, \mathcal{U}) + d(q, \mathcal{V}) \leq (1 - \chi(Q_A)) + (1 - \chi(Q_B)) = 2 - \chi(Q)$$

as required.

The case in which Q_A, Q_B are strictly essential is only a bit more difficult: Imagine coloring each component of Q_A (resp Q_B) that is not a ∂-parallel annulus red (resp blue). Since Q_A and Q_B are both essential, there are red and blue regions in $Q - P$. Since Q is connected there is a path in Q (possibly of length 0) with one end at a red region, one end at a blue region and no interior point in a colored region. Since the interior of the entire path lies in a collection of ∂-parallel annuli, it follows that the curves in $P \cap Q$ to which the ends of the path are incident are isotopic curves in P. Now apply the previous argument to a curve $q \subset P$ in that isotopy class of curves in P. □

4. SOBERING EXAMPLES OF LARGE DISTANCE

It is natural to ask whether Proposition 2.5 can, in any useful way, be extended to surfaces that are not essential. It appears to be unlikely. If one allows Q to be ∂-parallel, obvious counterexamples are easy: take a simple closed curve γ in N that is arbitrarily distant from $\mathcal{V}$ and use for Q a ∂-parallel annulus A constructed by pushing a regular neighborhood of γ slightly into M. Even if one rules out ∂-parallel surfaces but does allow Q to be compressible, a counterexample is obtained by tubing, say, a possibly knotted torus in M to an annulus A as just constructed.

On the other hand, it has been a recent theme in the study of embedded surfaces in 3-manifolds that, for many purposes, a connected separating surface Q in M will behave much like an incompressible surface if Q compresses to both sides, but not via disjoint disks. Would such a condition on Q be sufficient to guarantee the conclusion of Proposition 2.5? That is:

Question 4.1. *Suppose M is an irreducible compact orientable 3-manifold, and N is a compressible boundary component of M. Let $\mathcal{V}$ be the set of essential curves in N that bound disks in N. Suppose further that $(Q, \partial Q) \subset (M, \partial M)$ is a connected separating surface and q is any boundary component of Q. If Q is compressible into both complementary components, but not via disjoint disks, must it be true that $d(q, \mathcal{V}) \leq 1 - \chi(Q)$?*

In this section we show that there is an example for which the answer to Question 4.1 is no. More remarkably, the next section shows that it is the only type of bad example.

A bit of terminology will be useful. Regard ∂D^2 as the end-point union of two arcs, $\partial_\pm D^2$.

- Suppose $Q \subset M$ is a properly embedded surface and $\gamma \subset interior(M)$ is an embedded arc which is incident to Q precisely at $\partial\gamma$. There is a relative tubular neighborhood $\eta(\gamma) \cong \gamma \times D^2$ so that $\eta(\gamma)$ intersects Q precisely in the two disk fibers at the ends of γ. Then the surface obtained from Q by removing these two disks and attaching the cylinder $\gamma \times \partial D^2$ is said to be obtained by *tubing along* γ.
- Similarly, suppose $\gamma \subset \partial M$ is an embedded arc which is incident to ∂Q precisely in $\partial\gamma$. There is a relative tubular neighborhood $\eta(\gamma) \cong \gamma \times D^2$ so that $\eta(\gamma)$ intersects Q precisely in the two D^2 fibers at the ends of γ and $\eta(\gamma)$ intersects ∂M precisely in the rectangle $\gamma \times \partial_- D^2$. Then the properly embedded surface obtained from Q by removing the two D^2 fibers at the ends of γ and attaching the rectangle $\gamma \times \partial_+ D^2$ is said to be obtained by *tunneling along* γ.

Let P_0, P_1 be two connected compact subsurfaces in the same component N of ∂M, with each component of $\partial P_i, i = 0, 1$ essential in ∂M and $P_0 \subset interior(P_1)$. Let Q_1 be the properly embedded surface in M obtained by pushing P_1, rel ∂, into the interior of M. Let Q_0 denote the properly embedded surface obtained by pushing P_0 rel ∂ into the collar between P_1 and Q_1. Then the region R lying between Q_0 and Q_1 is naturally homeomorphic to $Q_1 \times I$. (Here $\partial Q_1 \times I$ can be thought of either as vertically crushed to $\partial Q_1 \subset \partial M$ or as constituting a small collar of ∂Q_1 in $P_1 \subset \partial M$.) Under the homeomorphism $R \cong Q_1 \times I$ the top of R (corresponding to $Q_1 \times \{1\}$) is Q_1 and the bottom of R (corresponding to $Q_1 \times \{0\}$) is the boundary union of Q_0 and $P_1 - P_0$. The properly embedded surface $Q_0 \cup Q_1 \subset M$ is called the *recessed collar* determined by $P_0 \subset P_1$ bounding R.

Recessed collars behave predictably under tunnelings:

Lemma 4.2. *Suppose $Q_0 \cup Q_1 \subset M$ is the recessed collar determined by $P_0 \subset P_1$, and $R \cong P_1 \times I$ is the component of $M - Q$ on whose boundary both Q_i lie. Let $\gamma \subset \partial M$ be a properly embedded arc in $\partial M - (Q_0 \cup Q_1)$. Let Q_+ be the surface obtained from $Q_0 \cup Q_1$ by tunneling along γ. Then*

1. *If $\gamma \subset (P_1 - P_0)$ and γ has both ends on ∂P_0 or if $\gamma \subset (\partial M - P_1)$, then Q_+ is a recessed collar.*
2. *If $\gamma \subset P_0$ then there is a compressing disk for Q_+ in $M - R$.*
3. *If $\gamma \subset (P_1 - P_0)$ and γ has one or both ends on ∂P_1, then there is a compressing disk for Q_+ in R.*

Proof. In the first case, tunneling is equivalent to just adding a band to either P_1 or P_0 and then constructing the recessed collar. In the second case, the disk $\gamma \times I$ in the collar between P_0 and Q_0 determines a compressing disk for Q_+ (that is, for the component of Q_+ coming from Q_0) that lies outside of R.

Similarly, in one of the third cases, when $\gamma \subset (P_1 - P_0)$ has both ends on ∂P_1, $\gamma \times I$ in the collar between P_1 and Q_1 determines a compressing disk for Q_+ (this time for the component of Q_+ coming from Q_1) that this time lies inside of R.

In the last case, when one end of $\gamma \subset (P_1 - P_0)$ lies on each of ∂P_0 and ∂P_1 a slightly more sophisticated construction is needed. After the tunneling construction, $\partial Q_+ \cap interior(P_1)$ has one arc component γ' that consists of two parallel copies of the spanning arc γ and a subarc of the component of ∂P_0 that is incident to γ. This arc $\gamma' \subset \partial Q_+$ can be pushed slightly into Q_+. Then the disk $\gamma' \times I$ (using the product structure on R) determines a compressing disk for Q_+ that lies in R. (The disk $\gamma' \times I$ looks much like the disk E in Figure 1.) □

One of the constructions of this lemma will be needed in a different context:

Lemma 4.3. *Suppose $Q_0 \cup Q_1 \subset M$ and $Q_1 \cup Q_2 \subset M$ are the recessed collars determined by connected surfaces $P_0 \subset interior(P_1)$ and $P_1 \subset interior(P_2)$. Let R_1, R_2 be the regions these recessed collars bound. Furthermore, let $\gamma_i \subset \partial M, i = 1, 2$ be properly embedded arcs spanning $P_1 - P_0$ and $P_2 - P_1$ respectively. That is, γ_i has one end point on each of $\partial P_i, \partial P_{i-1}$. Let Q_+ be the connected surface obtained from $Q_0 \cup Q_1 \cup Q_2$ by tunneling along both γ_1 and γ_2. Then either*

1. *There are disjoint compressing disks for Q_+ in R_1 and R_2 or*
2. *P_0 is an annulus parallel in P_1 to a component c of ∂P_1, and c is incident to both tunnels.*

In the latter case, Q_+ is properly isotopic to the surface obtained from the recessed collar $Q_1 \cup Q_2$ by tubing along an arc in interior(M) that is parallel to $\gamma_2 \subset \partial M$.

Proof. For P any surface with boundary, define an *eyeglass graph* in P to be the union of an essential simple closed curve in the interior of P and an embedded arc in the curve's complement, connecting the curve to ∂P.

Let $c_1 \subset \partial P_1$ and $c_0 \subset \partial P_0$ be the components to which the ends of γ_1 are incident. Let c_2 be the component of ∂P_1 (note: not ∂P_2) to which the end of γ_2 is incident. (Possibly $c_1 = c_2$.) Let α be any essential simple closed curve in P_0 and choose an embedded arc in $P_0 - \alpha$ connecting α to the end of γ_1 in c_0; the union of that arc, the closed curve α and the arc γ_1 is an eyeglass curve e_1 in P_1 which intersects $P_1 - P_0$ in the arc γ_1. Then the construction of Lemma 4.2, there applied to the eyeglass $\gamma_1 \cup c_0$, shows here that a neighborhood of the product $e_1 \times I \subset R_1 \cong P_1 \times I$ contains a compressing disk for Q_+ that lies in R_1 and which intersects Q_1 in a neighborhood of $e_1 \times \{1\}$.

Similarly, for β any essential simple closed curve in P_1, and an embedded arc in $P_1 - \beta$ connecting β to the end of γ_2 in c_2 we get an eyeglass $e_2 \subset P_2$ and a compressing disk for Q_+ that lies in R_2 and whose boundary intersects Q_1 only within a neighborhood of $e_2 \times \{1\}$. So if we can find disjoint such eyeglasses in P_1 and P_2 we will have constructed the required disjoint compressing disks.

Suppose first that P_0 is not an annulus parallel to c_1. Then P_0 contains an essential simple closed curve α that is not parallel to c_1. Since α is not parallel to c_1, no component of the complement $P_1 - e_1$ is a disk, so there is an essential simple closed curve β in the component of $P_1 - e_1$ that is incident to c_2. The same is true even if P_0 is an annulus parallel to c_1 so long as $c_1 \neq c_2$. This proves the enumerated items. See Figure 2

The proof that in case 2), Q_+ can be described by tubing Q_1 to Q_2 along an arc parallel to γ_2 is a pleasant exercise left to the reader. □

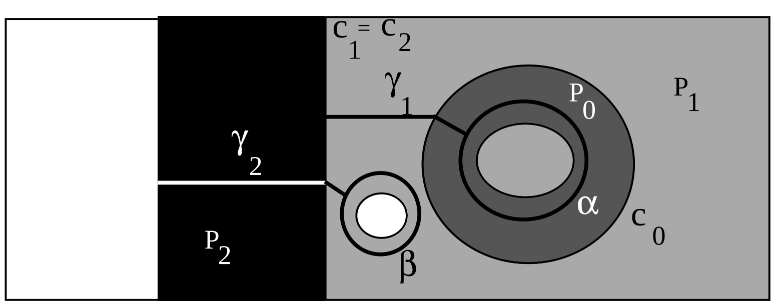

FIGURE 2

Now consider a particular type of tubing of a recessed collar. Suppose $Q_0 \cup Q_1 \subset M$ is the recessed collar bounding R determined by $P_0 \subset P_1 \subset \partial M$. Let ρ denote a vertical spanning arc in R, that is, the image in $R \cong P_1 \times I$ of $point \times I$, where $point \in P_0$. Let Q be the surface obtained from $Q_0 \cup Q_1$ by tubing along ρ. Then Q is called a *tube-spanned recessed collar*.

A tube-spanned recessed collar has nice properties:

Lemma 4.4. *Suppose Q is a tube-spanned recessed collar constructed as above. Then*

- *Q is connected and separating and Q compresses in both complementary components in M..*
- *If Q compresses in both complementary components via disjoint disks, then $P_1 \subset \partial M$ is compressible in M.*
- *If Q_+ is obtained from Q by tunneling, then either Q_+ is also a tube-spanned recessed collar or Q_+ compresses in both complementary components via disjoint disks. (Possibly both are true).*
- *If Q_+ is obtained from Q by tunneling together Q and a ∂-parallel connected incompressible surface Q', then either Q_+ is also a tube-spanned recessed collar or Q_+ compresses in both complementary components via disjoint disks. (Possibly both are true).*

Proof. The construction guarantees that Q is connected and separating. It compresses on both sides: Let Y denote the component $R - \eta(\rho)$ of $M - Q$ and let X be the other component. A disk fiber μ of $\eta(\rho)$ is a compressing disk for Q in X. To see a compressing disk for Q in Y, start with an essential simple closed curve in Q_0 containing the end of ρ in Q_0. The corresponding vertical annulus $A \subset R$ includes the vertical arc $\rho \subset R$. Then $A - \eta(\rho)$ is a disk in Y whose boundary is essential in Q.

To prove the second property, suppose that there are disjoint compressing disks, $D_X \subset X$ and $D_Y \subset Y$. ∂D_Y cannot be disjoint from the meridian μ of $\eta(\rho)$ since if it were, ∂D_Y would lie in either on the top or the bottom of $Y \cong (P_1 - point) \times I$, either of which is clearly incompressible in Y. So D_X cannot be parallel to μ. A standard innermost disk argument allows us to choose D_X so that $D_X \cap \mu$ contains no circles of intersection, and an isotopy of ∂D_X on Q ensures that any arc component of $\partial D_X - \mu$ is essential in one of the punctured surfaces $Q_1 \cap Q$ or $Q_0 \cap Q$. If D_X is disjoint from μ it lies on Q_1, say, but in any case it determines a compressing disk for P_1 in M, as required. If D_X is not disjoint from μ then an outermost disk in D_X cut off by μ would similarly determine a compression of P_1 in M.

The third property follows from Lemma 4.2. When the tunneling there leaves Q_+ as a recessed collar (option 1) then the operation here leaves Q_+ a tube-spanned recessed collar. If the tunneling arc γ lies in $P_1 - P_0$ and thereby gives rise to a compressing disk in R (option 3), the compressing disk D_Y there constructed lies in Y and so can clearly be kept disjoint from the vertical arc ρ. Then D_Y is disjoint from the compressing disk μ for X, as required. Finally, if γ lies in P_0 then the compressing disk D_X in $M - R$ constructed there lies in X and intersects Q_0 in a single essential arc. The simple closed curve in Q_0 from which A is constructed can be taken to intersect D_X in at most one point, so in the end the disk $D_Y \subset Y$ intersects D_X in at most one point. Then the boundary of a regular neighborhood of $\partial X \cup \partial Y$ in Q is a simple closed curve that bounds a disk in both X and Y, as required.

The fourth property is proven in a similar way. Suppose first that $\partial Q'$ is disjoint from P_1. If the region $P' \subset \partial M$ to which Q' is parallel is disjoint from P_1 then tunneling Q' to Q_1 just creates a larger ∂-parallel surface and Q_+ is a tube-spanned recessed collar. If $P_1 \subset P'$ then the region R' between Q' and Q_1 is a recessed collar and according to option 3 of Lemma 4.2 there is a compressing disk for Q_+ in $R' \cap X$ that is incident to Q_1 only in a collar of ∂Q_1. In particular it is disjoint from a compressing disk for Q in $R \cap Y$ constructed above from an annulus A that is incident to Q_1 away from this collar.

Next suppose that $\partial Q'$ lies in $P_1 - P_0$, so $P' \subset P_1 - P_0$. If the tunnel connects Q' to Q_0 then tunneling Q_0 to Q' just creates a larger ∂-parallel surface and Q_+ is a tube-spanned recessed collar. If the tunneling connects Q' to Q_1 then the argument is the same as when Q_+ is obtained from Q by tunneling into $P_1 - P_0$ with both ends of the tunnel on ∂P_1.

Finally suppose that $\partial Q'$ lies in P_0, so $P' \subset P_0$. Then the tunneling connects Q' to Q_0. The region R' between Q' and Q_0 is a recessed collar and according to option 3 of Lemma 4.2 there is a compressing disk for Q_+ in $R' \cap X$ that is incident to Q' only in a collar of $\partial Q'$. In particular it is disjoint from the compressing disk for Q in $R \cap Y$ constructed above from an annulus A incident to Q_0 in the image of $P' \subset P_1$ away from that collar. □

Corollary 4.5. *Suppose M is an irreducible compact orientable 3-manifold, and N is a compressible boundary component of M. Let $\mathcal{V}$ be the set of curves in N that arise as boundaries of compressing disks of N. Then for any $n \in \mathbb{N}$ there is a connected properly imbedded separating surface $(Q, \partial Q) \subset (M, N)$ so that Q compresses in both complementary components but not via disjoint disks and, for any component q of ∂Q, $d(q, \mathcal{V}) \geq n$.*

Proof. Let A_1 be an annulus in ∂M whose core has distance at least n from $\mathcal{V}$. Let $A_0 \subset A_1$ be a thinner subannulus and let Q be the tube-spanned recessed product in M that they determine. The result follows from the first two conclusions of Lemma 4.4.

□

5. ANY EXAMPLE IS A TUBE-SPANNED RECESSED COLLAR

It will be useful to expand the context beyond connected separating surfaces.

Definition 5.1. *Let $(Q, \partial Q) \subset (M, \partial M)$ be a properly embedded orientable surface in the orientable irreducible 3-manifold M. Q will be called a* splitting surface *if no component is closed, no component is a disk, and M is the union of two 3-manifolds X and Y along Q.*

We abbreviate by saying that Q splits M into the submanifolds X and Y.

The definition differs slightly from that of [JS, Definition 1.1], which allows Q to have closed components and disk components. Note also that the condition that M is the union of two 3-manifolds X and Y along Q is equivalent to saying that Q can be normally oriented so that any oriented arc in M transverse to Q alternately crosses Q in the direction consistent with the normal orientation and then against the normal orientation.

Definition 5.2. *Suppose as above that $(Q, \partial Q) \subset (M, \partial M)$ is a splitting surface that splits M into submanifolds X and Y. Q is bicompressible if both X and Y contain compressing disks for Q in M; Q is strongly compressible if there are such disks whose boundaries are disjoint in Q. If Q is not strongly compressible then it is weakly incompressible.*

Note that if Q is bicompressible but weakly incompressible ∂Q is necessarily essential in ∂M, for otherwise an innermost inessential component would bound a compressing disk for Q in $Y \cap \partial M$ (say). Such a disk, lying in ∂M, would necessarily be disjoint from any compressing disk for Q in X.

There are natural extensions of these ideas. One extension that will eventually prove useful is to ∂-compressions of splitting surfaces:

Definition 5.3. *A splitting surface $(Q, \partial Q) \subset (M, \partial M)$ is strongly ∂-compressible if there are ∂-compressing disks $D_X \subset X, D_Y \subset Y$ and $\partial D_X \cap \partial D_Y = \emptyset$.*

Here is our main result:

Theorem 5.4. *Suppose M is an irreducible compact orientable 3-manifold, N is a compressible boundary component of M and $(Q, \partial Q) \subset (M, \partial M)$ is a bicompressible, weakly incompressible splitting surface with a bicompressible component incident to N.*

Let $\mathcal{V}$ be the set of essential curves in N that bound disks in M and let q be any component of $\partial Q \cap N$. Then either

- *$d(q, \mathcal{V}) \leq 1 - \chi(Q)$ in the curve complex on N or*
- *q lies in the boundary of a ∂-parallel annulus component of Q or*
- *one component of Q is a tube-spanned recessed collar; all other components incident to N are incompressible and ∂-parallel.*

Note that in the last case, Q lies entirely in a collar of N.

Lemma 5.5. *Let $(Q,\partial Q)\subset(M,\partial M)$ be as in Theorem 5.4, splitting M into X and Y. Let Q_X be the result of maximally compressing Q into X. Then*

(1) *Q_X is incompressible in M and,*
(2) *there is a compressing disk D for N in M, so that some complete set of compressing disks for Q in X is disjoint from D and, moreover, $Q\cap D$ consists entirely of arcs that are essential in Q_X.*

Proof. First we show that Q_X is incompressible. This is in some sense a classical result, going back to Haken. A more modern view is in [CG]. Here we take the viewpoint first used in [ST, Prop. 2.2], which adapts well to other contexts we will need as well and is a good source for details missing here.

Q_X is obtained from Q by compressing into X. Dually, we can think of Q_X as a surface splitting M into X' and Y' (except possibly Q_X has some closed components) and Q is constructed from Q_X by tubing along a collection of arcs in Y'. Sliding one of these arcs over another or along Q_X merely moves Q by an isotopy, so an alternate view of the construction is this: There is a graph $\Gamma\subset Y'$, with all of its valence-one vertices on Q_X. A regular neighborhood of $Q_X\cup\Gamma$ has boundary consisting of a copy of Q_X and a copy of Q. (This construction of Q from Q_X could be called 1-surgery along the graph Γ.) The graph Γ may be varied by slides of edges along other edges or along Q_X; the effect on Q is merely to isotope it in the complement of Q_X.

Suppose that F is a compressing disk for Q_X in M. F must lie in Y', else Q could be further compressed into X. Choose a representation of Γ which minimizes $|F\cap\Gamma|$, and then choose a compressing disk E for Q in Y which minimizes $|F\cap E|$. If there are any closed components of $F\cap E$, an innermost one in E bounds a subdisk of E disjoint from F, Γ and Q; an isotopy of F will remove the intersection curve without raising $|F\cap\Gamma|$. So in fact there are no closed curves in $F\cap E$.

The disk F must intersect the graph Γ else F would lie entirely in Y and so be a compressing disk for Q in Y that is disjoint from compressing disks of Q in X. This would contradict the weak incompressiblity of Q. One can view the intersection of $\Gamma\cup E$ with F as a graph $\Lambda\subset F$ whose vertices are the points $\Gamma\cap F$ and whose edges are the arcs $F\cap E$.

If there is an isolated vertex of the graph $\Lambda\subset F$ (i. e. a point in $\Gamma\cap F$ that is disjoint from E) then the vertex would correspond to a compressing disk for Q in X which is disjoint from E, contradicting weak irreducibility. If there is a loop in $\Lambda\subset F$ whose interior contains no vertex, an innermost such loop would bound a subdisk of F that could be used to simplify E; that is to find disk E_0 for Q in Y so that $|F\cap E_0|<|F\cap E|$, again a contradiction.

We conclude that Λ has a vertex w that is incident to edges but to no loops of Λ. Choose an arc β which is outermost in E among all arcs of $F \cap E$ which are incident to w. Then β cuts off from E a disk E' with $E' - \beta$ disjoint from w. Let e be the edge of Γ which contains w. Then the disk E' gives instructions about how to isotope and slide the edge e until w and possibly other points of $\Gamma \cap F$ are removed, lowering $|\Gamma \cap F|$, a contradiction that establishes the first claim.

To establish the second claim, first note that by shrinking a complete set of compressing disks for Q in X very small, we can of course make them disjoint from any D; the difficulty is ensuring that $Q_X \cap D$ then has no simple closed curves of intersection.

Choose D and isotope Q_X to minimize the number of components $|D \cap Q_X|$, then choose a representation of Γ which minimizes $|D \cap \Gamma|$, and finally then choose a compressing disk E for Q in Y which minimizes $|D \cap E|$. If there are any closed components of $D \cap E$, an innermost one in E bounds a subdisk of E disjoint from D, Γ and Q; an isotopy of D will remove the intersection curve without raising either $|D \cap Q_X|$ or $|D \cap \Gamma|$. So in fact there are no closed curves in $D \cap E$.

Suppose there are closed curves in $D \cap Q_X$. An innermost one in D will bound a subdisk D_0. Since Q_X is incompressible, ∂D_0 also bounds a disk in Q_X; the curve of intersection could then be removed by an isotopy of Q_X, a contradiction.

From this contradiction we deduce that all components of $D \cap Q_X$ are arcs. All arcs are essential in Q_X else $|D \cap Q_X|$ could be lowered by rechoosing D. The only other components of $D \cap Q$ are closed curves, compressible in X, each corresponding to a point in $D \cap \Gamma$. So it suffices to show that $D \cap \Gamma = \emptyset$. The proof is analogous to the proof of the first claim, where it was shown that Γ must be disjoint from any compressing disk F for Q_X in Y', but now for F we take a (disk) component of $D - Q_X$.

If no component of $D - Q_X$ intersects Γ there is nothing to prove, so let F be a component intersecting Γ and regard $\Lambda = (\Gamma \cup E) \cap F$ as a graph in F, with possibly some edges incident to the arcs $Q_X \cap D$ lying in ∂F. As above, no vertex of Λ (i. e. point of $\Gamma \cap F$) can be isolated in Λ and an innermost inessential loop in Λ would allow an improvement in E so as to reduce $D \cap E$. Hence there is a vertex w of Λ that is incident to edges but no loops in Λ. An edge in Λ that is outermost in E among all edges incident to w will cut off a disk from E that provides instructions how to slide the edge e of Γ containing w so as to remove the intersection point w and possibly other intersection points. As in the first case, some sliding of the end of e may necessarily be along arcs in Q_X, as well as over other edges in Γ. □

Proof of Theorem 5.4: Just as in the proof of Proposition 2.5 the proof is by induction on $1-\chi(Q)$. Since Q contains no disk components, $1-\chi(Q) \geq 1$.

If compressing disks for Q were incident to two different components of Q, then there would be compressing disks on opposite sides incident to two different components of Q, violating weak incompressibility. So we deduce that all compressing disks for Q are incident to at most one component Q_0 of Q. Q_0 cannot be an annulus, else the boundaries of compressing disks in X and Y would be parallel in Q_0 and so could be made disjoint. If Q also contains an essential component Q' incident to N then $1-\chi(Q') \leq 1-\chi(Q-Q_0) < 1-\chi(Q)$ and so, by Proposition 2.5, for any component q' of $\partial Q' \cap N$, $d(q', \mathcal{V}) \leq 1-\chi(Q') < 1-\chi(Q)$. This implies that $d(q, \mathcal{V}) \leq d(q', \mathcal{V}) + d(q, q') \leq 1-\chi(Q)$ as required. So we will also henceforth assume that no component of Q incident to N is essential.

We can also assume that each component of $Q-Q_0$ is itself an incompressible surface. For suppose D is a compressing disk for a component $Q_1 \neq Q_0$ of Q, chosen among all such disks to have a minimal number of intersection components with Q. If the interior of D were disjoint from Q then D would be a compressing disk for Q itself, violating weak incompressibility as described above. Similarly, an innermost circle of $Q \cap D$ in D must lie in Q_0. Consider a subdisk D' of D (possibly all of D) with the property that its boundary is second-innermost among components of $D \cap Q$. That is, the interior of D' intersects Q exactly in innermost circles of intersection, each bounding disks in X, say. If $\partial D'$ is not in Q_0 then it is also a compressing disk for Q_X, contradicting the first statement in Lemma 5.5. The argument is only a bit more subtle when $\partial D'$ is in Q_0, cf the No Nesting Lemma [Sc2, Lemma 2.2].

Let Q_- be the union of components of Q that are not incident to N. Since Q_- is incompressible, each compressing disk for N is disjoint from Q_-. In particular, it suffices to work inside the 3-manifold $M-\eta(Q_-)$ instead of M. So, with no loss of generality, we can assume that $Q_- = \emptyset$, ie each component of Q is incident to N.

Since each component of Q other than Q_0 is incompressible and not essential, each is boundary parallel. In particular, removing one of these components Q_1 from Q still leaves a bicompressible, weakly incompressible splitting surface, though each component of $M-Q_1$ in the region of parallelism between Q_1 and ∂M would need to be switched from X to Y or vice versa. Since we don't care about the boundaries of ∂-parallel annuli, all such components can be removed from Q without affecting the hypotheses or conclusion. If there remains a ∂-parallel component Q_1 that is not an annulus, then consider $Q' = Q-Q_1$. We have $1-\chi(Q') < 1-\chi(Q)$ so the inductive hypothesis applies. Then either Q_0 is a tube-spanned recessed product (and we are done) or for any component q' of $\partial Q'$, $d(q', \mathcal{V}) \leq 1 - \chi(Q') < 1 - \chi(Q)$. This implies that $d(q, \mathcal{V}) \leq d(q', \mathcal{V}) + d(q, q') \leq 1 - \chi(Q)$ and again we are done. So we may as well assume that $Q = Q_0$ is connected and, as we have seen, not an annulus.

Claim: The theorem holds if Q is strongly ∂-compressible.

Proof of claim: Suppose there are disjoint ∂-compressing disks $F_X \subset X$, $F_Y \subset Y$ for Q in M. Let Q_x, Q_y denote the surfaces obtained from Q by ∂-compressing Q along F_X and F_Y respectively, and let Q_- denote the surface obtained by ∂-compressing along both disks simultaneously. (We use lower case x, y to distinguish these from the surfaces Q_X, Q_Y obtained by *maximally compressing* Q into respectively X or Y.) A standard innermost disk, outermost arc argument between F_x and a compressing disk for Q in X shows that Q_x is compressible in X. Similarly, Q_y is compressible in Y.

Each of Q_x, Q_y has at most two components, since Q is connected. Suppose that Q_x (say) is itself bicompressible. If it were strongly compressible, the same strong compression pair of disks would strongly compress Q, so we conclude that the inductive hypothesis applies to Q_x, so we apply the theorem to Q_x. One possibility is that one component of Q_x is a tube-spanned recessed collar and the other (if there are two components) is ∂-parallel. But by Lemma 4.4 this case implies that Q is also a tube-spanned recessed collar and we are done. The other possibility is that for q_x a component of the boundary of an essential component of Q_x, $d(q_x, \mathcal{V}) \leq 1 - \chi(Q_x) < 1 - \chi(Q)$. This implies that $d(q, \mathcal{V}) \leq d(q_x, \mathcal{V}) + d(q, q_x) \leq 1 - \chi(Q)$ and again we are done. So we henceforth assume that Q_x (resp Q_y) is compressible into X (resp Y) but not into Y (resp X).

It follows that Q_- is incompressible, for if Q_- is compressible into Y, say, then such a compressing disk would be unaffected by the tunneling that recovers Q_x from Q_- and Q_x would also compress into Y.

On the other hand, if Q_- is essential in M then the claim follows from Proposition 2.5. So the only remaining case to consider in the proof of the claim is when Q_- is incompressible and not essential, so all its components are ∂-parallel. Since Q is connected, Q_- has at most three components. Suppose there are exactly three Q_0, Q_1, Q_2. If the three are nested (that is, they can be arranged as Q_0, Q_1, Q_2 are in Lemma 4.3) then that lemma shows that the weakly incompressible Q must be a tube-spanned recessed collar, as required. If no pairs of the three components of Q_- are nested, then Q itself would be boundary parallel and so could not be compressible on the side towards N. Finally, suppose that two components (Q_0, Q_1, say) are nested, that Q_2 is ∂-parallel in their complement, and Q_x, say, is obtained from Q_1, Q_2 by tunneling between Q_1 and Q_2, so Q_x is ∂-parallel. Q_x is also compressible; the compressing disk either also lies in a collar of N, or, via the parallelism to the boundary, the disk represents a compressing disk D for N in M whose boundary is disjoint from ∂Q_x. In the latter case we have, for q_x any component of ∂Q_x, $d(q_x, \partial D) \leq 1$. Then for q any component of Q, $d(q, \partial D) \leq d(q_x, \partial D) + d(q, q_x) \leq 2 \leq 1 - \chi(Q)$ and we are done. The former case can only arise if there are boundary components of Q_1 and Q_2 that cobound an annulus, and that annulus is spanned by the tunnel. Moreover, since a resulting compressing disk for Q_x lies in N and so cannot persist into Q, the tunnel attaching Q_0 must be incident to that same boundary component of Q_1. It is easy to see then that Q is a tube-spanned recessed product, where the two recessed surfaces are Q_0 and the union of Q_1, Q_2 along their parallel boundary components.

Similar arguments apply if Q_- has one or two components. This completes the proof of the Claim.

Compressing a surface does not affect its boundary, so the theorem follows immediately from Lemma 5.5 and Proposition 2.5 unless the surface Q_X, obtained by maximally compressing Q into X has the property that each of its non-closed components is boundary parallel in M. Of course the symmetric statement holds also for the surface Q_Y obtained by maximally compressing Q into Y; indeed, all the ensuing arguments would apply symmetrically to Q_Y simply by switching labels X and Y throughout. So henceforth assume that all components of Q_X are either closed or ∂-parallel. There are some of the latter, since Q has boundary.

Let Q_0 be an outermost ∂-parallel component of Q_X that is not closed. That is Q_0 is a component which is parallel to a subsurface of ∂M and no component of Q_X lies in the region $R \cong Q_0 \times I$ of parallelism. As in the proof of Lemma 5.5, use the notation $X' \subset X$ and $Y' \supset Y$ for the two 3-manifolds into which Q_X splits M, noting that, unlike for Q, some components of Q_X may be closed. Note also that $\Gamma \subset Y'$.

Case 1: Some such outermost region R lies in Y'

In this case the other side of Q_0 lies in X', and so its interior is disjoint from Γ. Since Q is connected, this implies that all of Q lies in R. In particular, $\Gamma \subset R$, all compressing disks for Q in Y also lie in R, and $Q_0 = Q_X$. Let $D \subset M$ be a ∂-reducing disk for M as in Lemma 5.5 so that Γ is disjoint from D and $D \cap Q_0$ consists only of arcs that are essential in Q_0.

Any outermost such arc in D cuts off a ∂-reducing disk $D_0 \subset D$. Suppose first that D_0 lies in $M - R$ and let Q_0' be the surface created from Q_0 by ∂-compressing along D_0. By Lemma 2.3 Q_0' is incompressible, so all boundary components of Q_0' are essential in ∂M unless Q_0 is an annulus that is parallel to ∂M also via $M - R$. The latter would imply that Q_0 is a longitudinal annulus of a solid torus, D is a meridian of that solid torus and we could have taken for D_0 the half of D that does lie in R. In the general case, the union of D_0 with a disk of parallelism in R gives a ∂-reducing disk for M that is disjoint from $\partial Q'_0$ so for any boundary component q' of Q'_0, $d(q', \mathcal{V}) \leq 1$. Then for q any component of $\partial Q = \partial Q_X = \partial Q_0$, $d(q, \mathcal{V} \leq d(q', \mathcal{V}) + d(q, q') \leq 2 \leq 1 - \chi(Q)$ and we are done. In any case, we may as well then assume that D_0 lies in $R \subset Y'$.

Since Γ is disjoint from D_0, D_0 is a ∂-reducing disk for Q as well, lying in Y. Then a standard outermost arc argument in D_0 shows that a compressing disk for Q in Y can be disjoint from D_0. Then ∂-reducing Q along D_0 leaves a surface that is still bicompressible (for meridians of Γ constitute compressing disks in X) but with $1 - \chi(Q)$ reduced. The proof then follows by induction. (In fact, this argument can be enhanced to show directly that Case 1 simply cannot arise.)

It remains to consider the case in which all outermost components of Q_X are ∂-parallel via a region that lies in X'. We distinguish two further cases:

Case 2: There is nesting among the non-closed components of Q_X. We will prove then that Q must be a tube-spanned recessed collar.

In this case, let Q_1 be a component that is not closed (so it is ∂-parallel) and is “second-outermost”. That is, the region of parallelism between Q_1 and ∂M contains in its interior only outermost components of Q_X; denote the union of the latter components by Q_0. Then the region between Q_0 and Q_1 is itself a product $R \cong Q_1 \times I$ but one end contains Q_0 as a possibly disconnected subsurface. Since outermost components cut off regions lying in X', $R \subset Y'$. We now argue much as in Case 1: Since $\Gamma \subset Y'$ and Q is connected, all of Γ must lie in R, so $Q_X = Q_1 \cup Q_0$. Let D be a ∂-reducing disk for M that is disjoint from Γ and intersects Q_X only in arcs that are essential in Q_X. As in Case 1, each outermost arc of $D \cap Q_X$ in D lies in Q_0.

Choose a complete collection of ∂-compressing disks $\mathcal{F}$, in the region of parallelism between Q_1 and ∂M, so that the complement $Q_1 - \mathcal{F}$ is a single disk D_Q. Each disk in $\mathcal{F}$ is incident to Q_1 in a single arc. Now import the argument of Lemma 5.5 into this context: Let E be a compressing disk for Y, here chosen so that $E \cap \mathcal{F}$ is minimized. This means first of all that $E \cap \mathcal{F}$ is a collection of arcs. As in the proof of Lemma 5.5, Γ may be slid and isotoped so it is disjoint from $\mathcal{F}$. Γ is incident to Q_1 since Q is connected. Since D_Q is connected, the ends of Γ on D_Q may be slid within D_Q so that ultimately Γ is incident to D_Q in a single point. ∂E is necessarily incident to that end, since Q is weakly incompressible. It follows that ∂E cannot be incident to Q only in D_Q (else ∂E could be pushed off the end of Γ in D_Q) so ∂E must intersect the arcs $\partial \mathcal{F} \cap Q_1$. Let $\beta \subset (\mathcal{F} \cap E)$ be outermost in E among all arcs incident to components of $\partial \mathcal{F} \cap Q_1$. Let E_0 be the disk that β cuts off from E.

If both ends of β were in $\mathcal{F} \cap Q_1$ then, since each disk of $\mathcal{F}$ is incident to Q_1 in a single arc, β would cut off a subdisk of $\mathcal{F}$ that could be used to alter E, creating a compressing disk for Y that intersects $\mathcal{F}$ in fewer points. We conclude that the other end of β is on Q_0. Since β is outermost among those arcs of $E \cap \mathcal{F}$ incident to D_Q, ∂E_0 traverses the end of Γ on D_Q exactly once. So, as in the proof of Lemma 5.5, it can be used to slide and isotope an edge ρ of Γ until it coincides with β. Hence the edge $\rho \subset \Gamma$ can be made into a vertical arc (i. e. an I-fiber) in the product structure $R = Q_1 \times I$.

Using that product structure and an essential circle in the component of Q_0 that is incident to ρ, ρ can be viewed as part of a vertical incompressible annulus A with ends on Q_1 and Q_0. Now apply the argument of Lemma 5.5 again: $A - \rho$ is a disk E'. Since E' is a disk, use the argument of Lemma 5.5 to slide and isotope the edges of $\Gamma - \rho$ until they are disjoint from E'. After these slides, E' is revealed as a compressing disk for Q in Y. On the other hand, if there is in fact any edge γ in $\Gamma - \rho$, the compressing disk for Q in X given by the meridian of $\eta(\gamma)$ would be disjoint from E, contradicting weak incompressiblilty of Q. So we conclude that in fact $\Gamma = \rho$ and so, other than the components of Q_X incident to the ends of ρ, each component of Q_X is a component of Q; since Q is connected, there are no such other components. That is, Q is obtained by tubing Q_1 to the connected Q_0 along ρ and so is a tube-spanned recessed collar. This completes the argument in this case.

Case 3: All non-closed components of Q_X are outermost among the components of Q_X. We will show that in this case Q is strongly ∂-compressible; the proof then follows from the Claim above.

We have already seen that all non-closed components of Q_X are ∂-parallel through X'. Choose a ∂-reducing disk $D \subset M$ as in Lemma 5.5 so that D is disjoint from the graph Γ, intersects Q_X mimimally and intersects Q only in arcs that are essential in Q_X. Although there is no nesting among the components of Q_X, it is not immediately clear that the arcs $D \cap Q_X$ are not nested in D. However, it is true that each outermost arc cuts off a subdisk of D that lies in X', as shown in the proof of Case 1 above. In what follows, D' will represent either D, if no arcs of $D \cap Q_X$ are nested in D, or a disk cut off by a "second-outermost" arc of intersection λ_0 if there is nesting. Let $\Lambda \subset D'$ denote the collection of arcs $D' \cap Q$; one of these arcs (namely λ_0) may be on $\partial D'$.

Consider how a compressing disk E for Q in Y intersects D'. All closed curves in $D' \cap E$ can be removed by a standard innermost disk argument redefining E. Any arc in $D' \cap E$ must have its ends on Λ; a standard outermost arc argument can be used to remove any that have both ends on the same component of Λ. If any component of $\Lambda - \lambda_0$ is disjoint from all the arcs $D' \cap E$, then Q could be ∂-compressed without affecting E. This reduces $1 - \chi(Q)$ without affecting bicompressibility, so we would be done by induction. Hence we restrict to the case in which each arc component of $\Lambda - \lambda_0$ is incident to some arc components of $D' \cap E$. See Figure 3.

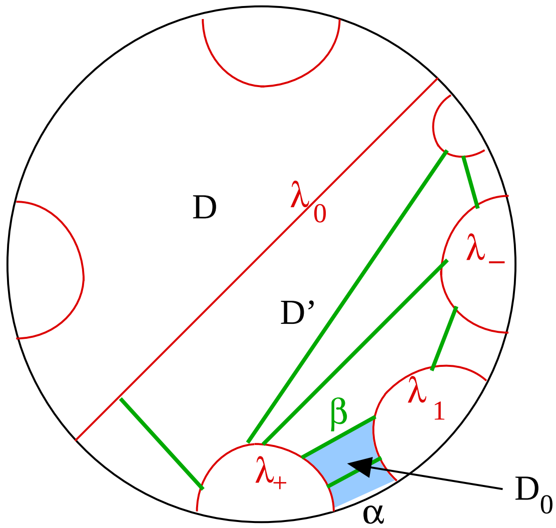

FIGURE 3

It follows that there is at least one component $\lambda_1 \neq \lambda_0$ of Λ with this property: any arc of $D' \cap E$ that has one end incident to λ_1 has its other end incident to one of the (at most two) neighboring components $\lambda_\pm$ of Λ along $\partial D'$. (Possibly one or both of $\lambda_\pm$ are λ_0.) Let β be the outermost arc in E among all arcs of $D' \cap E$ that are incident to the special arc λ_1. We then know that the other end of β is incident to (say) λ_+ and that the disk $E_0 \subset E$ cut off by β from E, although it may be incident to D in its interior, at least no arc of intersection $D \cap interior(E_0)$ is incident to λ_1.

Let D_0 be the rectangle in D whose sides consist of subarcs of λ_1, λ_+, ∂D and all of β. Although E may intersect this rectangle, our choice of β as outermost among arcs of $D \cap E$ incident to λ_1 guarantees that E_0 is disjoint from the interior of D_0 and so is incident to it only in the arc β. The union of E_0, D_0 along β is a disk $D_1 \subset Y$ whose boundary consists of the arc $\alpha = \partial M \cap \partial D_0$ and an arc $\beta' \subset Q$. The latter arc is the union of the two arcs $D_0 \cap Q$ and the arc $E_0 \cap Q$. If β' is essential in Q, then D_1 is a ∂-compressing disk for Q in Y that is disjoint from the boundary compressing disk in X cut off by λ_1. So if β' is essential then Q is strongly ∂-compressible and we are done by the Claim.

Suppose finally that β' is inessential in Q. Then β' is parallel to an arc on ∂Q and so, via this parallelism, the disk D_1 is itself parallel to a disk D' that is disjoint from Q and either is ∂-parallel in M or is itself a ∂-reducing disk for M. If D' is a ∂-reducing disk for M, then $\partial D' \in \mathcal{V}$, $d(\mathcal{Q}, \mathcal{V}) \leq 1$ and we are done. On the other hand, if D' is parallel to a subdisk of ∂M, then an outermost arc of ∂D in that disk (possibly the arc α itself) can be removed by an isotopy of ∂D, lowering $|D \cap Q| = |D \cap Q_X|$. This contradiction to our original choice of D completes the proof. □

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