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Local detection of strongly irreducible Heegaard splittings

Martin Scharlemann [1,2]

University of California, Department of Mathematics, Santa Barbara, CA 93106-3080, USA

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Abstract

Let S be a Heegaard splitting surface of a compact orientable 3-manifold M. If S is strongly irreducible, the manner in which it can intersect a ball or a solid torus in M is very constrained and the allowable configurations are simple and useful. Splitting surfaces not conforming to these simple local pictures must be weakly reducible.

Keywords: Compression body; Heegaard splitting; Strongly irreducible

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0. Introduction

The study of Heegaard splittings of compact orientable 3-manifolds has become dramatically more useful since Casson and Gordon introduced the notion of *strongly irreducible* splittings [2]. All irreducible splittings of non-Haken 3-manifolds are also strongly irreducible [2] and any irreducible splitting can be decomposed into a series of strongly irreducible splittings [7].

It has sometimes been useful to understand how such splittings can intersect simple 3-dimensional submanifolds, e.g., a ball [5, 3.7] or a solid torus [5, Section 2]. In general the intersection can be quite complicated, and, without side conditions, it seems to be impossible to distinguish strongly irreducible from weakly reducible or even reducible splittings by their local behavior, e.g., how they intersect a ball. Here we show that, with quite reasonable side-conditions, the picture changes dramatically and the "local" picture of a strongly irreducible splitting is very structured. The structure we describe here for the intersection with a ball (2.1) is new, though inspired by the partial result

[1] Supported in part by a National Science Foundation grant.
[2] E-mail: mgscharl@math.ucsb.edu.

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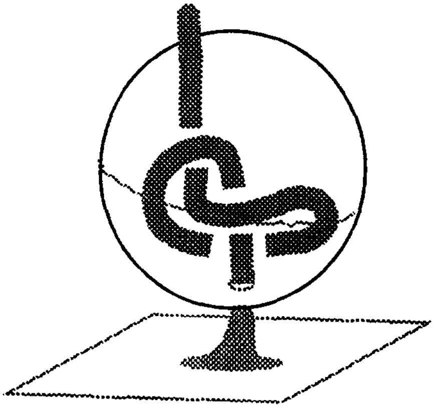

Fig. 1a.

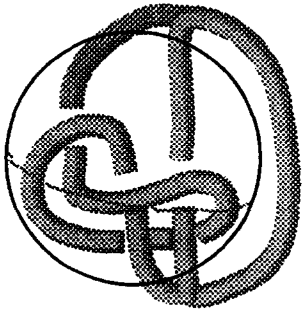

Fig. 1b.

of [5, 3.2]. The structure for the intersection with a solid torus (3.3) was previously derived in [5, Section 2], but we offer a more efficient proof. Conversations with Abby Thompson were particularly useful for Section 3.

As motivating examples, consider two examples of how a surface might intersect a 3-ball in a 3-manifold M.

Example 1. Let S be any properly imbedded surface in a 3-manifold M (e.g., a splitting surface for M) and B be a disjoint 3-ball. Let τ be a knotted arc in B and $\partial\eta(\tau)$ the corresponding knotted tube inside the 3-ball. Let α be an arc in $M-(B\cup S)$ that has one end on S and the other end on a point of $\partial\tau \subset \partial B$. Use $\alpha\cup\tau$ to make a "finger-push" on S through B (see Fig. 1a).

Afterwards, $B\cap S$ is the knotted tube $\partial\eta(\tau)$. This process can be repeated to make very complicated tube intersections of S with B. Note that, after this construction, $\partial B - S$ is necessarily compressible in the complement of S.

Example 2. Let Γ be a graph in S^3 intersecting the ball B as shown in Fig. 1b. Then (via arc slides) it is easy to see that $S = \partial\eta(\Gamma)$ is a standard genus two Heegaard splitting of S^3. $S\cap B$ is quite complicated (and could be made more so) yet, in this example, $\partial B - S$ is incompressible in the complement of S. Of course, S is not a strongly irreducible splitting, but it is not obvious how this global fact affects the local structure $B\cap S$. We will show that it does, and in a very dramatic way, once we rule out the previous sort of example by requiring that $\partial B - S$ be incompressible in the complement of S.

1. Preliminaries

Notation. Let $|Q|$ denote the number of components of Q, typically a compact 0 or 1-manifold. For X a complex in a manifold, let $\eta(X)$ denote a regular neighborhood of X in the manifold.

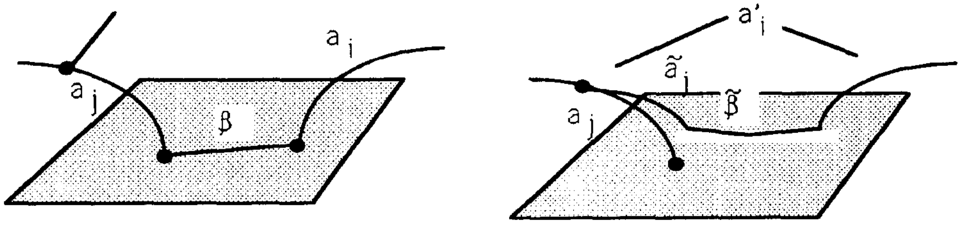

Fig. 2.

For Γ a finite 1-complex in a 3-manifold M, let $\partial\Gamma$ be the set of valence one vertices. For Q a properly imbedded surface in M (or $Q \subset \partial M$) we say Γ is *properly imbedded* in the complement of Q if $\Gamma \cap Q = \partial\Gamma$. Let a_i and a_j, $i \neq j$, be edges of Γ with ends on the same vertex. An edge slide of a_i over a_j replaces a_i with $a_i' = a_i \cup \widetilde{a}_j$, where $\widetilde{a}_j$ is a copy of a_j pushed to be slightly disjoint from a_j. Similarly, if the edges of a_i and a_j each have an end on the same component of $Q \subset M$ and β is an arc imbedded in Q connecting the ends, an arc slide of a_i over a_j replaces a_i with $a_i' = a_i \cup \widetilde{\beta} \cup \widetilde{a}_j$, where $\widetilde{a}_j$ and $\widetilde{\beta}$ are copies of a_j and β pushed to be slightly disjoint from $\Gamma \cup Q$ (see Fig. 2). In general, any path β on the boundary of a regular neighborhood of $\Gamma \cup Q$ which begins at an end of an edge a_i but otherwise never crosses that end of a_i defines a series of edge slides of a_i. Just deform the part of β lying on Γ to be an edge path, and then regard β as a series of edges and arcs in Q over which a_i is slid. This replaces a_i with the union of a_i and a copy of β pushed slightly away from $\Gamma \cup Q$. A complex obtained from Γ by a series of such slides is called *slide equivalent* to Γ.

A *compression body* H is constructed by adding 2-handles to a (surface) $\times I$ along a collection of disjoint simple closed curves on (surface) $\times \{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component (surface) $\times \{1\}$ of ∂H is denoted $\partial_+ H$ and the surface $\partial H - \partial_+ H$, which may or may not be connected, is denoted $\partial_- H$. If $\partial_- H = \emptyset$ then H is a *handlebody*. If $H = \partial_+ H \times I$, H is called a *trivial* compression body.

Let S be a closed connected orientable surface imbedded in an orientable 3-manifold M. S is a *splitting surface* for a Heegaard splitting if S divides M into two compression bodies H_1 and H_2 with $\partial_+ H_1 = S = \partial_+ H_2$. An *elementary stabilization* of S is the splitting surface obtained by taking the connected sum of pairs $(M, S)\#(S^3, T)$, for T the standard unknotted torus in S^3. A Heegaard splitting is *stabilized* if it is an elementary stabilization of another splitting. This is equivalent to the existence of proper disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $\partial D_1 \cap \partial D_2$ a single point in S.

The Heegaard splitting is *reducible* if there is an essential simple closed curve $c \subset S$ which bounds imbedded disks in both H_1 and H_2. A Heegaard splitting is *weakly reducible* ("not strongly irreducible" in [2]) if there exist essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $\partial D_1 \cap \partial D_2 = \emptyset$. If H is reducible it is clearly weakly reducible.

We briefly collect some well-known facts:

Proposition 1.1. *A stabilized Heegaard splitting S with* genus$(S) > 1$ *is reducible.*

Proof. Let c be the boundary of a regular neighborhood of $\partial D_1 \cup \partial D_2$ in S. Since genus$(S) > 1$, c is essential. □

Proposition 1.2. *A reducible Heegaard splitting of an irreducible* 3*-manifold is stabilized.*

Proof. Heegaard splittings of S^3 are unique [8] (see also [6]). □

Proposition 1.3. *Any Heegaard splitting of a* ∂*-reducible* 3*-manifold into two nontrivial compression bodies is weakly reducible.*

Proof. This is immediate from [2, 1.1]. □

Proposition 1.4. *Any Heegaard splitting of a reducible* 3*-manifold is reducible.*

Proof. This is essentially [4]. □

Proposition 1.5. *Let Γ be the spine of a compression body defined by a Heegaard splitting of a* 3*-manifold $M \neq S^3$. If a cycle in Γ is contained in some* 3*-ball then the splitting is reducible.*

Proof. The proof, due to Frohman [3], is a clever application of Proposition 1.4. The cycle must contain an edge of Γ but otherwise may contain arcs in $\partial_- H$. □

2. Local detection: how splitting surfaces intersect balls

Definition. Suppose $(Q, \partial Q) \subset (B, \partial B)$ is a connected properly imbedded planar surface in a 3-ball and $|\partial Q| = m$. Let $\Gamma \subset B$ be the cone on m points in ∂B. Then Q is *unknotted* if it is properly isotopic to $\partial\eta(\Gamma)$. Equivalently, there is a unique nondisk component P of $\partial B - Q$ and Q is parallel to P in B.

The goal of the present section is the proof of the following precise characterization:

Theorem 2.1. *Suppose $H_1 \cup_S H_2$ is a strongly irreducible Heegaard splitting of an orientable* 3*-manifold M and B is a ball in M. Let T_i be the planar surface $\partial B \cap H_i$ properly imbedded in H_i, and suppose each T_i is incompressible in H_i. Then $S \cap B$ is connected, planar, and unknotted in B.*

Lemma 2.2 (No nesting). *Suppose $H_1 \cup_S H_2$ is a strongly irreducible Heegaard splitting of a 3-manifold M and F is a disk in M transverse to S with $\partial F \subset S$. Then ∂F also bounds a disk in some H_i.*

Proof. The proof is by induction on $|S \cap \mathrm{int}(F)|$. If the interior of F is disjoint from S there is nothing to prove. If $S - F$ has any disk components D then, by replacing the subdisk of F bounded by ∂D by a parallel copy of D we can decrease $|S \cap \text{int}(F)|$. So assume that each curve in $S \cap F$ is essential in S.

A disk component of $F - S$ compresses S in one of the two handlebodies, say H_1. Then by strong irreducibility of S, all disk components of $F - S$ lie in H_1. If any pair of curves of $F \cap S$ are nested then the outer curve of the innermost such pair cuts off a component P of $F - S$ so that all but one of the curves in ∂P are adjacent to disks in H_1 (hence $P \subset H_2$) and precisely one, denoted α, is not. Compress S into H_1 along 2-handles whose cores are the disks with boundaries on ∂P. Let M_- be the 2-manifold obtained from H_2 by attaching these 2-handles to H_2. Then $\alpha \subset \partial M_-$ is inessential in M_- so, by strong irreducibility and Proposition 1.3, α is inessential in ∂M_-. Push the disk α bounds in ∂M_- slightly into H_1 and observe that this is then a disk D in H_1 whose boundary is parallel to α in the component of F adjacent to P across α. Replacing the subdisk of F bounded by α (or all of F if $\alpha = \partial F$) with D lowers $|S \cap \text{int}(F)|$. □

Corollary 2.3. *Under the hypotheses of Theorem* 2.1, *at most one component P of $\partial B - S$ is not a disk.*

Proof. The alternative would give a planar surface Q all but one of whose boundary components bound disks in $\partial B - S$. Then the union of Q with those disks would be a disk in M whose boundary q is essential in $\partial B - S$ but, by the no-nesting lemma, inessential in, say, H_1. This contradicts the assumption in Theorem 2.1 that T_1 is incompressible in H_1. □

Proof of Theorem 2.1. Proceed by induction on the number $m > 0$ of curves in the boundary of $S \cap M$. If $m = 1$ then, essentially by [8], $S_B = S \cap B$ is a disk as required. So assume the theorem is true for $|\partial S_B| < m$.

Let $P \subset H_2$ be the unique planar component of $\partial B - S$ identified by Corollary 2.3. Since P is incompressible in H_2, P does not lie in a 3-ball in H_2 and so must intersect any complete set of meridian disks of H_2. It follows that P is ∂-compressible in H_2. Let D be a ∂-compressing disk, so $\partial D = \alpha \cup \beta$ where α is an essential arc in P and β is a (necessarily essential) arc in $S - \partial P$. There are four cases to consider, the last two of which exploit the fact that M is necessarily irreducible by Proposition 1.4:

Case 1: $D \subset B$ and the ends of α lie on distinct components of ∂P.

In this case, use D to isotope $\beta \subset S$ across P, changing S_B to S' with $|\partial S'| = m - 1$. The hypotheses of the theorem are still satisfied, since the only nondisk surface in $\partial B - S'$ is just $P - \eta(\alpha) \subset P$, so S' is unknotted in B. That is, S' divides B into two components, and $H_2 \cap B \cong S' \times I$. S_B can be recovered from S' by tunneling across a disk component of $H_1 \cap \partial B$ and so is also unknotted (see Fig. 3).

Case 2: $D \subset M - B$ and the ends of α lie on distinct components of ∂P.

Again use D to isotope $\beta \subset S$ across P, changing S_B to an unknotted surface S' in B. Dually, there is a ∂-compression of S' across $H_1 \cap B$ that converts S' into S_B. The ∂-compressing disk D' intersects S' in an arc γ. In S the ∂-compression across D

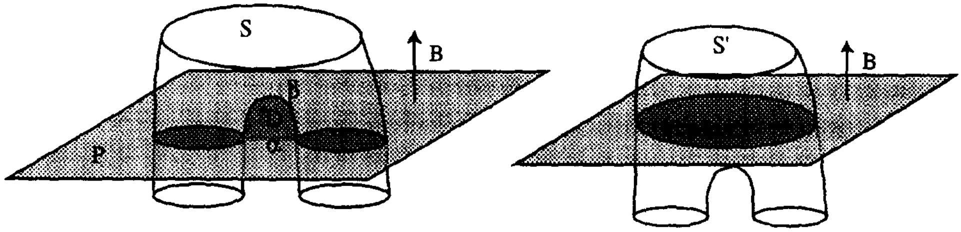

Fig. 3.

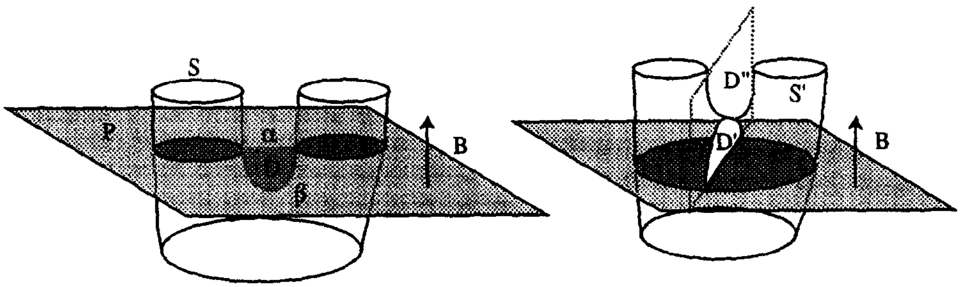

Fig. 4.

changes $S \cap \partial B$ by banding together two circle components along β and γ is the cocore of this band, intersecting β in a single point.

Since S' is unknotted, the component of $B - S'$ that lies in H_2 is homeomorphic to $S' \times I$. In particular, there is a disk $D'' \subset B \cap H_2$ whose boundary is the union of γ and arc in $\partial B - S$. But the ∂-compression across D' which changes S' to S_B stretches D'' across all of B, turning it into a compressing disk for $\partial B - S$ (see Fig. 4). The contradiction shows that this case does not arise.

Case 3: $D \subset B$ and both arcs of α lie on the same curve c in ∂P.

A ∂-compression on D changes S_B to S' with $|\partial S'| = m + 1$ and converts the disk in $\partial B \cap H_1$ which c bounds into an annulus component A of $\partial B \cap H_1$. As in Case 2, there is a dual arc γ to β in S, this time lying outside B, and having one end on each end of A. We cannot immediately apply the inductive hypothesis, but Lemma 2.3 implies that A must be compressible in H_1.

A cannot compress in $H_1 - B$, for otherwise the sphere comprised of this compressing disk and a compressing disk for A in B would be a nonseparating, hence reducing, 2-sphere. (It intersects the union of γ and a spanning arc of A in a single point.) So A compresses in $H_1 \cap B$. The compressing disk divides B into two balls and in each, by inductive hypothesis, S' is standard. It follows that S_B is obtained by ∂-summing two unknotted surfaces (along β) and so is unknotted. (See Fig. 5.)

Case 4: $D \subset M - B$ and both ends of α lie on the same curve c of ∂P.

As in Case 3, a ∂-compression along D converts S_B into S' with $|\partial S'| = m + 1$ and creates an annular component A of $\partial B - S'$. This time A must compress in $H_1 - B$, by the same reasoning as in Case 3 but now with $\gamma \subset B$. Let $E \subset H_1 - B$ be a compressing

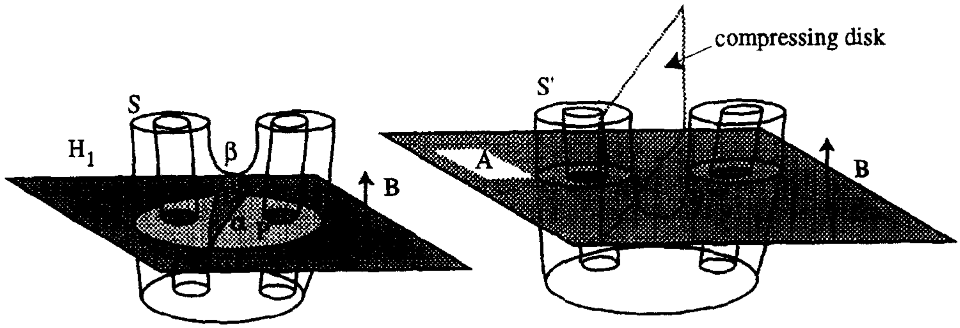

Fig. 5.

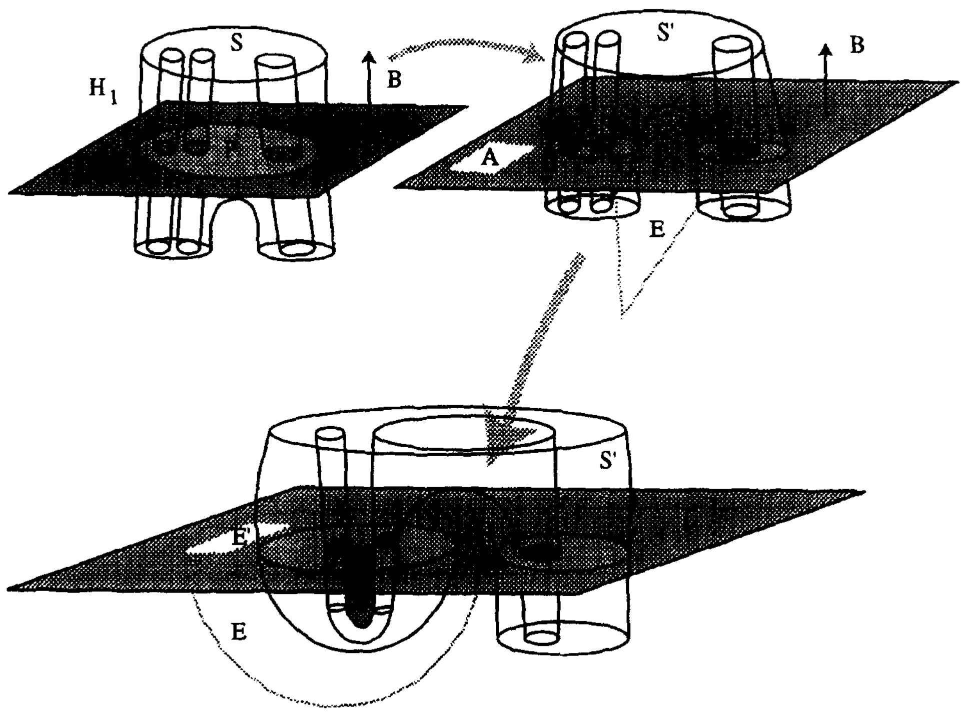

Fig. 6.

disk. ∂E divides ∂B into two hemispheres. Since M is irreducible, the union of E and one of the hemispheres E' bounds a ball B' not containing the other hemisphere. Now since E is transverse to A, the ∂-compression of S_B to S' is just an isotopy on $S \cap B'$. In particular, $S \cap B'$ satisfies the hypotheses of the theorem and the inductive hypotheses, so $S \cap B'$ is unknotted. It follows that there is a ∂-compression of $S \cap B'$ to $\partial B'$ via a disk in $H_2 \cap B' \subset H_2 - B$ whose boundary intersects distinct components of $S \cap \partial B'$. (See Fig. 6.) This returns us to Case 2. □

3. Knot neighborhoods: how splitting surfaces intersect solid tori

To characterize intersections with more complicated submanifolds, it will be useful to have an analogue to weak reducibility for general surfaces with boundary.

Definition. A properly imbedded oriented surface $(Q, \partial Q) \subset (M, \partial M)$ is a *splitting surface* if M is the union of two 3-manifolds X and Y along Q so that ∂X induces the given orientation on Q and ∂Y induces the opposite orientation. A compressing disk for Q in X (respectively Y) is called a *meridian* disk in X (respectively Y) and its boundary a meridian curve for X (respectively Y).

The splitting surface Q is called *strongly compressible* if there are meridian disks in X and Y with disjoint boundaries. Otherwise Q is called *weakly incompressible*. Q is *stabilized* if there are meridian disks in X and Y whose boundaries meet at a single point in Q. A stabilized splitting surface (not an unknotted torus) is clearly strongly compressible.

Proposition 3.1. *Suppose V is a solid torus and $(Q, \partial Q) \subset (V, \partial V)$ is a weakly incompressible splitting surface for V, splitting V into X and Y, and ∂Q is a collection of nonmeridinal essential curves in ∂V. Suppose that no component of Q is an annulus, and there are ∂-compressing disks $D_X \subset X$ and $D_Y \subset Y$ for Q with $\partial D_X \cap \partial D_Y = \emptyset$. Then Q is the surface obtained from one or two incompressible annuli in V by attaching a tube parallel to an arc in ∂V.*

Proof. Let Q_X, Q_Y, Q_0 be the surfaces obtained from Q by, respectively, ∂-compression along D_X, D_Y and both D_X and D_Y simultaneously. The ∂-compression along D_X changes an annulus component of $\partial V - Q$ into a disk in X, and the boundary of the disk is essential in Q_X since no component of Q is an annulus. It follows from weak incompressibility that Q_X cannot be compressed into Y, so neither can $Q_0 \subset Q_X$. Symmetrically Q_0 cannot be compressed into X, so Q_0 is a collection of annuli and disks. There can be no disks on grounds of Euler characteristic, since Q contains no annuli and has an even number of boundary components. Thus Q is built from a collection of incompressible annuli by tunneling on two arcs γ_X (dual to D_X) and γ_Y (dual to D_Y) in $\partial V - Q_0$.

Since $\partial V - Q$ consists of annuli, there is a component q_0 of ∂Q_0 incident to at least one end of both γ_X and γ_Y. If both γ_X and γ_Y are inessential arcs in $\partial V - Q_0$ then it is easy to see that Q would be stabilized. If γ_X is essential and γ_Y is not (or vice versa), then the end of γ_X at q_0 must lie between the ends of γ_Y, or else ∂Q would not be essential in ∂V. It is easy to reinterpret this construction as connecting one or two annuli via a ∂-parallel tube (the meridian of the tube is the disk component of $\partial V - Q_X$ and the arc in ∂V is $\partial V \cap D_X$). (See Fig. 7.) If both γ_X and γ_Y are essential, but γ_X (or γ_Y) has both ends on the same annulus of Q_0, then Q would be stabilized. Just by parity, the ends of γ_X and γ_Y not on q_0 cannot lie on the same annulus in Q_0, so the ends of γ_X and γ_Y lie on three distinct annuli, at least two of which then must be parallel in V. It follows that Q is obtained by tubing together two annuli (one of which can be obtained

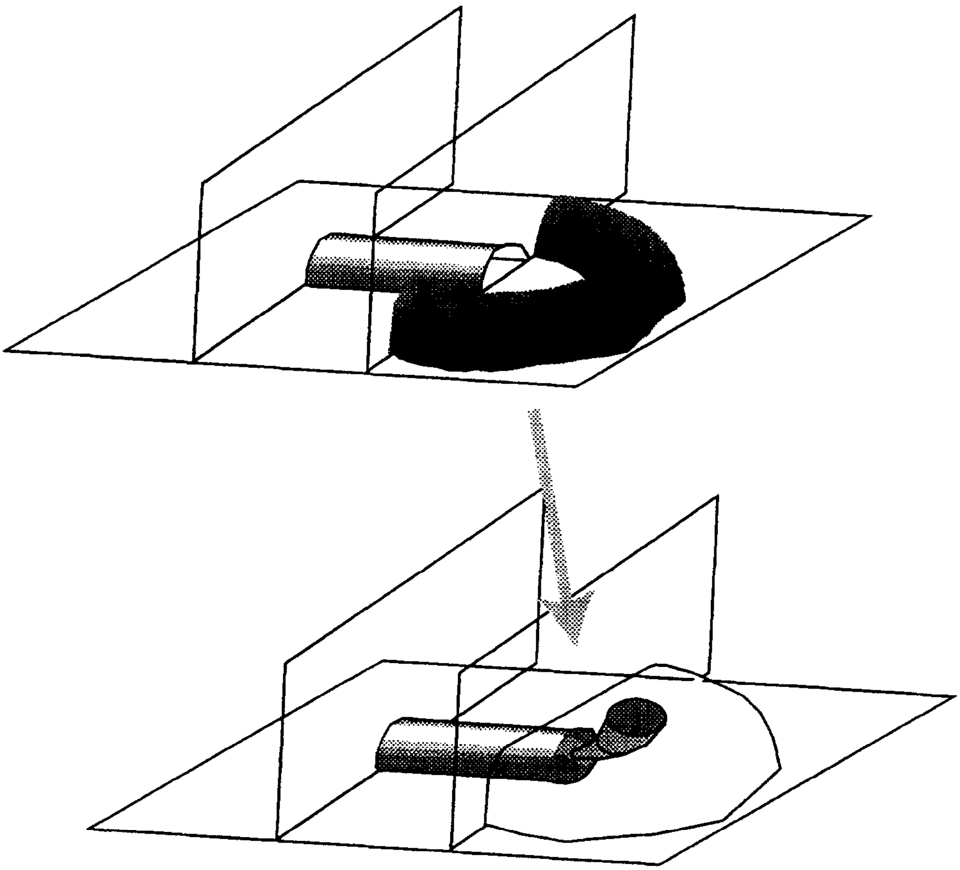

Fig. 7.

from Q_0 by tunneling together two parallel annuli in Q_0 and then compressing). (See Fig. 8.) □

Proposition 3.2. *Suppose V is a solid torus and Q is a weakly incompressible splitting surface for V splitting V into X and Y. Suppose that $\partial V - Q$ is a collection of non-meridinal annuli and suppose Q is compressible in both X and Y. Then Q is the union of some incompressible annuli and one other component Q_0. Q_0 is obtained from one or two incompressible annuli by attaching a tube parallel to an arc in ∂V.*

Proof. Nothing is lost by removing all annuli components from Q: remove an annulus and, in one of its two complementary components, switch X to Y and vice versa. The result will still satisfy the hypothesis. So assume Q has no annuli and let A be the surface obtained from Q by maximally compressing into X. A divides V into the remnants X' of X and a 3-manifold W obtained from Y by attaching some 2-handles in X. It follows from strong irreducibility and [7, 2.2] (or, implicitly, [2]) that A cannot be compressed in W and, by construction, cannot be compressed in X. So A is a collection

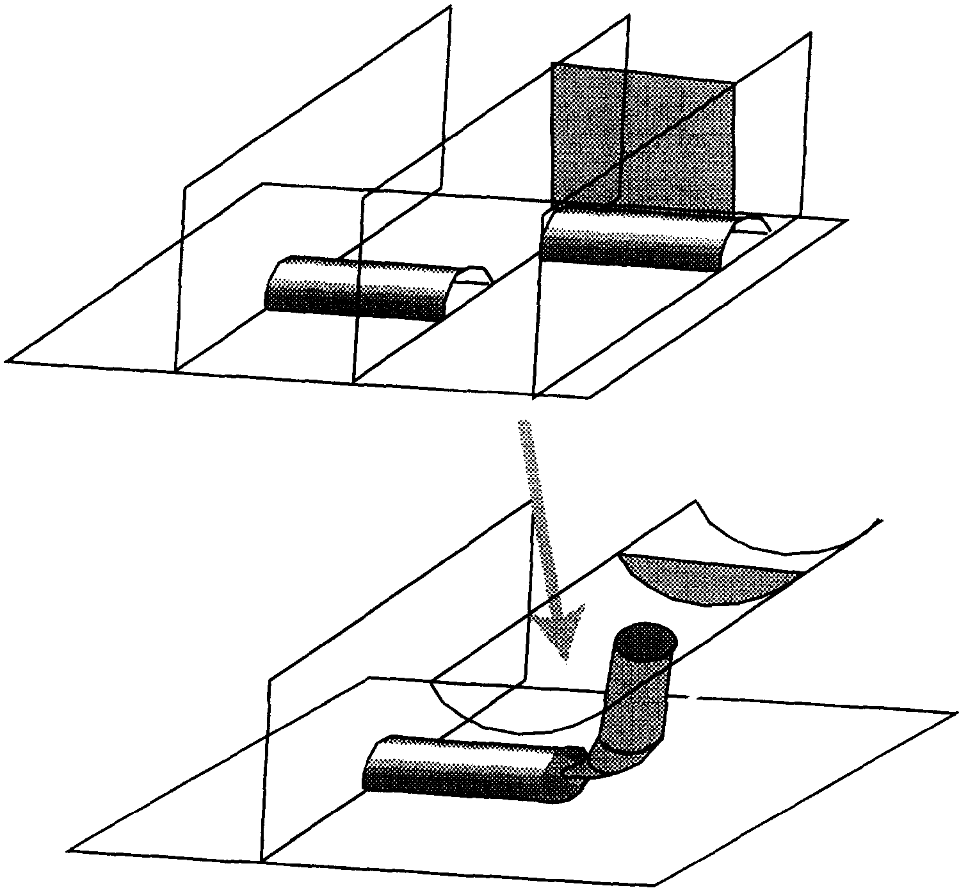

Fig. 8.

of incompressible annuli (hence the notation A). The original Q is obtained from A by attaching tubes along a collection Γ of proper arcs in W.

Let D be a meridian disk for V chosen to minimize $|A \cap D|$, so that D intersects each annular component of A and each annular component of $\partial V - A$ in proper spanning arcs. $V - \eta(D)$ is a ball on whose boundary lie two copies $D_\pm$ of D. $A - \eta(D)$ is a collection of disks, in fact rectangles with two sides on ∂V and one side on each of $D_\pm$.

Now slide and isotope Γ to minimize $|\Gamma \cap D|$. The argument of [6, Section 2] guarantees that in fact $|\Gamma \cap D| = \emptyset$, that is, all of Γ can be pushed off of D. Recall the argument: Minimize $\Gamma \cap D$ by handle slides and broken handle slides (see [6]) and let E be the hypothesized compressing disk in Y. According to [6, 2.2] there is, among any remaining points in $\Gamma \cap D$, at least one point for which the corresponding meridian disk in X is disjoint from ∂E. This contradicts weak incompressibility, so we conclude that in fact Γ is disjoint from D.

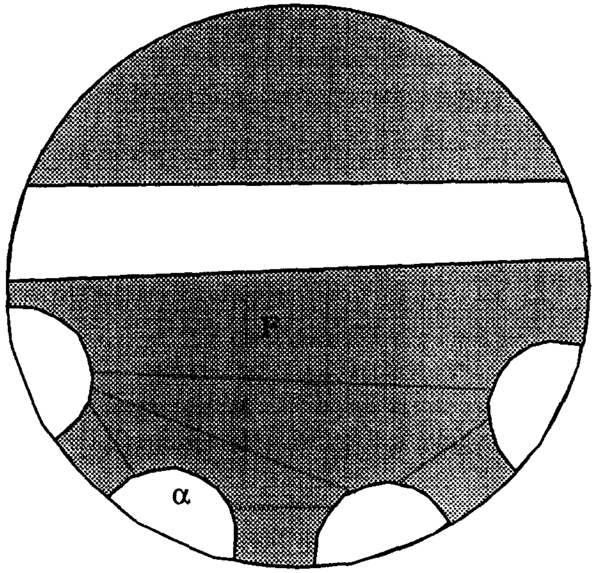

Fig. 9.

Once $\Gamma \cap D = \emptyset$, minimize $E \cap D$ over all possible compressing disks for Q in Y. If $E \cap D = \emptyset$ then $(A - \eta(D)) \cup \Gamma$ contains a cycle lying in a ball, contradicting Proposition 1.5. So we can assume $E \cap D$ is nonempty and, by a simple innermost disk, outermost arc argument, is in fact a collection of proper arcs lying in the disks $D - \eta(A)$ so that no arc has both ends abutting the same component of $D \cap A$.

Claim 3.2.1. *Every arc of $A \cap D$ that is outermost in D cuts off a disk lying in X'.*

Proof. Let α be such an arc. If the disk it cuts off from D lies in W then, since every annulus A must be incident to some edge in Γ, some edges of Γ lie in the solid torus W_0 between the annulus A_0 containing α and ∂V. Since these edges are disjoint from D they (and $A_0 - D$) form a cycle in the ball $W_0 - D$, contradicting Proposition 1.5. We conclude that the disk α cuts off from D lies in X'. We can also conclude that $\partial E \cap \alpha \neq \emptyset$ for otherwise we could ∂-compress Q at α, converting the annulus of $\partial V - Q$ on whose ends α lies into a meridian disk for Q whose boundary is disjoint from ∂E, contradicting weak incompressibility of Q. (See Fig. 9.) □

Call two arcs of $D \cap A$ *adjacent* if there is an arc of $\partial D - Q$ which has one end on each of them.

Claim 3.2.2. *There is an outermost arc α in $A \cap D$ with the property that any arc of $E \cap D$ which has one end on α has its other end on an adjacent component of $A \cap D$.*

Proof. Since all outermost arcs of $A \cap D$ in D cut off disks lying in X', a simple argument shows there is a disk component F of $D - \eta(A)$ so that all but at most one arc of $\partial F \cap A$ is outermost in D. Again from Proposition 1.5 it follows that $E \cap F \neq \emptyset$. A simple outermost arc argument in F shows that there is at least one arc α of $\partial F \cap A$, outermost in D, with the property that *every* arc of $E \cap F$ incident to α has its other end on an adjacent arc of $\partial F \cap A$ in F. □

We will now apply the idea behind [6, 2.2] to the arc α. Among all arcs of $E \cap D$ that are adjacent to α, let δ be one that is outermost in E. Then δ cuts off from E a subdisk E_0 whose boundary consists of δ and an arc $\beta \subset \partial E$ whose interior is disjoint from α. Since δ has an end on α, it follows from the definition of α that the other end of δ is on an adjacent arc of $\partial F \cap A$ and so it is parallel to an arc δ' in $\partial V - Q$. Moreover, since δ is outermost in E among arcs incident to α, the rectangle R between δ and δ' in F is disjoint from E_0. Then $E_0 \cup R$ is a ∂-compressing disk for Q in W that never crosses (a slightly adjusted) α. The proof follows from Proposition 3.1.

Theorem 3.3. *Suppose $H_1 \cup_S H_2$ is a strongly irreducible Heegaard splitting of a 3-manifold M and $V \subset M$ is a solid torus such that ∂V intersects S in parallel essential nonmeridinal curves. Then S intersects V in a collection of ∂-parallel annuli and possibly one other component, obtained from one or two annuli by attaching a tube along an arc parallel to a subarc of ∂V. If the latter sort of component is in V, then $S - V$ is incompressible in $M - V$.*

Proof. Among all possible counterexamples to the theorem, choose one which minimizes $|S \cap \partial V|$.

Claim. *No annulus F in $\partial V - S$ is parallel to a subannulus of S.*

Proof. An isotopy of this subannulus A_S of S across F will reduce $|S \cap \partial V|$. If after the isotopy $S \cap \partial V \neq \emptyset$ we are done by induction. If after the isotopy $S \cap \partial V = \emptyset$ and A_S was in V then, before the isotopy, $S_V = A_S$ and so was not a counterexample. If A_S was in $M - V$ then after the isotopy $S \subset V$, so either S lies in a ball or $M - V \subset H_i$ is also a solid torus and M is a Lens space. The former contradicts strong irreducibility of S and the latter would imply that S is a torus [1] and so S_V is a collection of essential annuli, as required. □

If $S_V = S \cap V$ is incompressible in V then it is a union of annuli and there is nothing to prove. So suppose S_V compresses into H_1, say, in V. Consider the collection of annuli $F = H_2 \cap \partial V \subset H_2$. First note that F is incompressible in H_2, for a compressing disk could not lie in V since F is nonmeridinal in ∂V, nor could it lie outside for this would again imply that M is a Lens space. Hence F is ∂-compressible in H_2. If a ∂-compressing disk lies in $H_2 - V$ then, since there are no ∂-parallel annuli in $\partial V - S$, the result is a meridian disk for H_2 in $M - V$. This, together with a meridian disk for H_1 in V, contradicts strong irreducibility. Hence any ∂-compressing disk lies in $H_2 \cap V$.

It follows that S_V is an irreducible weakly incompressible surface in V that compresses into both H_1 and H_2. The result now follows from Proposition 3.2.

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