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THE FOUR-DIMENSIONAL SCHOENFLIES CONJECTURE IS TRUE FOR GENUS TWO IMBEDDINGS

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IT WAS established by Brown[2] that any locally-flat imbedding of S^{n-1} in S^n divides S^n into two domains, each of whose closures is an n-ball. Somewhat later[5] the h-cobordism theorem further established that if S^{n-1} is a smooth of PL submanifold of S^n then so are the resulting n-balls, provided that $n \geq 5$. (The case $n \leq 3$ had been known since Alexander[1].)

For $n = 4$ little is known. The goal of this paper is to present an elementary proof of the conjecture for the special case described below.

A collared handlebody decomposition of a 3-manifold M will be a decomposition $\emptyset = W_0' \subset W_1 \subset W_1' \subset W_2 \subset W_2' \subset \ldots \subset W_{n-1} \subset W_{n-1}' \subset W_n \simeq M$ such that, for $0 < i \leq n$, W_i is obtained from W_{i-1}' by attaching a handle $h_i \simeq D^k \times D^{3-k}$ to $\partial W_{i-1}'$ along $\partial D^k \times D^{3-k}$, and W_i' is obtained from W_i by attaching a collar to ∂W_i.

It will be convenient to regard S^4 as the two-point compactification of $S^3 \times \mathbb{R}$, so $S^3 \times \mathbb{R} \subset S^4$. Let $p: S^3 \times \mathbb{R} \to \mathbb{R}$, $\pi: S^3 \times \mathbb{R} \to S^3$ be the standard projections. A PL imbedding $g: S^3 \to S^3 \times \mathbb{R} \subset S^4$ is a *critical level imbedding* if there is a collared handlebody decomposition $W_1 \subset W_1' \subset \ldots \subset W_n$ of $g(S^3)$ and a collection $t_1 < \ldots < t_n$ for $p|g(S^3)$ such that, for $1 \leq i \leq n$, $h_i = (S^3 \times \{t_i\}) \cap g(S^3)$ and, for $t_i < t < t_{i+1}$, $\pi((S^3 \times \{t\}) \cap g(S^3)) = \pi(g(\partial W_i))$. Thus the handles of $g(S^3)$ are all horizontal in $S^3 \times \mathbb{R}$ and the collars are all vertical. It is known that any PL imbedding is PL isotopic to a critical level imbedding[3].

A critical level imbedding $g: S^3 \to S^4$ is said to have *genus n* if the associated handlebody decomposition of S^3 has k 0-handles and $(k + n - 1)$ 1-handles. The goal here is to show that any PL critical level imbedding $g: S^3 \to S^4$ of genus ≤ 2 is standard, that is, it is PL ambient isotopic to the standard imbedding.

We work in the PL category exclusively; all maps, imbeddings and isotopies are understood to be PL.

§1. RESTRICTED 0–1 HANDLE CANCELLATION

First some notation. A handle of dimension n and index k (a k-handle) is a PL homeomorph of $D^k \times D^{n-k}$. The disks $D^k \times \{0\}$ and $\{0\} \times D^{n-k}$ are called, respectively, the core and the cocore of the handle.

For $g: S^3 \to S^3 \times \mathbb{R} \subset S^4$ a critical level imbedding we will let $S_J^3 = S^3 \times J$ for J an interval in $\mathbb{R}$, $W_t = g(S^3) \cap S^3_{(-\infty, t]}$, and $M_t = \partial W_t \subset S_t^3$. The handles $\{h_1, \ldots, h_n\}$ of the associated handlebody decomposition of $g(S^3)$ will always be listed in order of ascending critical value $\{t_1, \ldots, t_n\}$.

We denote the core of h_i by c_i and the cocore by c_i'. Note that for $t_i \leq t < t_{i+1}$, $\pi(c_i')$ and $\pi(c_{i+1})$ are imbedded disks in the complement of $\pi(M_t) \subset S^3$.

LEMMA 1.1. *If there are ambient isotopies, with $\pi(M_t)$ invariant, of $\pi(c_i')$ and $\pi(c_{i+1})$ to*

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disjoint disks in S^3, then g is isotopic to a critical level imbedding with the same critical levels and handlebody structure, except $h_{i+1} \subset S^3_{t_i}$ and $h_i \subset S^3_{t_{i+1}}$.

COROLLARY 1.2. *If h_{i+1} has lower index than h_i, the critical levels on which they occur may be switched.*

The proofs of both are elementary.

Following 1.2 we may isotope any critical level imbedding $g: S^4 \to S^4$ to one in which the handles appear in order of increasing index. Such an imbedding is called *rectified.*

Suppose a rectified imbedding has 0-handles $h_1, \ldots, h_k$ and $t_k \leq t < t_{k+1}$. Then $\pi(M_t)$ consists of a collection of 2-spheres $\{\pi(\partial h_i)\}$ in S^3, each dividing S^3 into two components, called the interior ($= \pi(h_i^0)$) and the exterior of the 0-handle. Note that if $\pi(\partial h')$ is in the interior of $\pi(\partial h)$ then h' must occur at a higher critical level than h; in particular the interior of $\pi(h')$ must be in the interior of $\pi(h)$. We say in this case that $h \lesssim h'$; then $\lesssim$ is a partial order on the 0-handles consistent with the total order given by the height at which the handles appear in $S^3 \times \mathbb{R}$.

LEMMA 1.3. *Suppose $g: S^3 \to S^4$ is a rectified imbedding with 0-handles $h_1, \ldots, h_n$ appearing at levels $t_1 < \ldots . < t_n$. For any 0-handle h_j and $t > t_j$ there is an isotopy of g with support in W_t to a rectified imbedding g' with critical levels $t_1 < \ldots < t_n$ and handles $h'_1 < \ldots < h'_n$ such that*

(a) *$h_i = h'_i$ for $i > j$*

(b) *There is a permutation $\sigma \in S(j)$ such that $\pi(\partial h_i) = \pi(\partial h'_{\sigma(i)})$ for $i \leq j$*

(c) *For exactly the i such that $h_i \lesssim h_j$, the interior of $\pi(h'_{\sigma(i)})$ is the exterior of $\pi(h_i)$ and vice versa.*

Proof. Suppose h_j is minimal under $\lesssim$, so $\pi(\partial h_j)$ lies in the interior of the projection of no other 0-handle. Then by 1.1, g can be isotoped so that h_j has the lowest critical level, t_1. But the 4-ball lying below $S^3_{t_1}$ is then an isotopy rel M_{t_1} between the component of $S^3_{t_1}$ lying over the interior and exterior of $\pi(\partial h_j)$.

The proof then follows by induction on the number of handles $h_i \lesssim h_j$. Apply the previous argument to the minimal 0-handle such that $h_i \lesssim h_j$, thereby reducing the number of handles such that $h_i \lesssim h_j$. □

Suppose $g: S^3 \to S^4$ is a rectified imbedding with 0-handles $h_1, \ldots, h_k$. Then by general position it is easy to arrange that for any 1-handle $h \simeq (D^1 \times D^2)$ of $g(S^3)$ the attaching disks $\partial D^1 \times D^2$ project under π into $\pi(\partial h_1) \cup \ldots \cup (\partial h_k)$. If they lie on the projection of distinct 0-handles, h is called a *connecting* 1-handle.

LEMMA 1.4. (0–1 *handle cancellation*) *Suppose $g: S^3 \to S^4$ is a rectified imbedding with 0-handles $h_1, \ldots, h_k$ and 1-handles $h_{k+1}, \ldots, h_l$ and h_i and h_j are a* 0 *and* 1*-handle respectively such that*

(a) *a single attaching disk for h_j projects into $\pi(\partial h_i)$.*

(b) *No 1-handle with critical value less than t_j has an attaching disk projecting into $\pi(\partial h_i)$. Then g is isotopic to a rectified imbedding with one fewer 0-handle and one fewer 1-handle.*

Proof. By 1.3 we can assume that $\pi(h_j)$ lies in the exterior of $\pi(\partial h_i)$. By 1.1 there is, for $t_j < t < t_{j+1}$ an isotopy of W_t, fixed on M_t, to a critical level imbedding in which the handles $\{h_1, \ldots, h_j\}$ appear in the following order: First $\{h_1, \ldots, h_{i-1}\}$ then those handles of $\{h_{i+1}, \ldots, h_{j-1}\}$ which project to the exterior of $\pi(\partial h_i)$, then h_i, then h_j, then those handles in $\{h_{i+1}, \dots, h_{j-1}\}$ which project to the interior of ∂h_i. Now isotope so that h_i and h_j occur at the same level. Then the union of h_i and h_j can be "tilted" slightly and viewed as an isotopy rel boundary of the attaching disk of h_j not contained in h_i. This isotopy can then be made vertical as in [3] (by incorporating $h_i \cup h_j$ into or deleting it from the 0-handle to which h_j attaches h_i) to produce a critical level imbedding with two fewer handles. Now use 1.1 to rectify the imbedding. □

PROPOSITION 1.5. *A critical level imbedding $g: S^3 \to S^4$ of genus n is isotopic to a critical level imbedding with k 0-handles and $(k + n - 1)$-handles, $k \leq n$.*

Proof. All but the requirement that $k \leq n$ is true by definition of genus. Since the union of the 0-handles and 1-handles is connected, there must be at least $(k - 1)$ connecting 1-handles. Then if $k > n$ there is a 0-handle to which only connecting 1-handles are attached. Rectify $g(S^3)$ and apply 1.4 to reduce k. □

§2. REMARKS ON THE GENERAL CASE

The *dual* of a critical level imbedding $g: S^3 \to S^3 \times \mathbb{R} \subset S^4$ is the critical level imbedding obtained by reflecting S^4 through $S^3 \times \{0\}$. All the i-handles of $g(S^3)$ become $3 - i$ handles of its dual.

The critical level imbedding $g: S^3 \to S^4$ is called *compact* if a component Y of $S^4 - g(S^3)$ is contained in $S^3 \times \mathbb{R}$. Let X denote the other component and $Y_t = Y \cap S^3_t$, $X_t = X \cap S^3_t$. Any critical level imbedding g may be made compact (if it isn't already) merely by applying 1.3 to the lowest 0-handle in the dual of g, that is, the highest 3-handle in g.

A handle h_i attached at the level t_i is called an inside (resp. outside) handle if, for $t_{i-1} < t < t_i$, $\pi(h_i)$ lies in $\pi(Y_t)$ (resp. $\pi(X_t)$). Furthermore, by applying 1.3 to the lowest 0-handle and, dually, to the highest 3-handle we can switch the inside and outside, so simultaneously all inside handles become outside and vice versa.

2.1 *Remark.* Suppose $t < t_i < t'$ and h_i is an inside handle. Then $Y \cap S^3_{(-\infty, t]}$ is *PL* homeomorphic to $Y \cap S^3_{(-\infty, t']}$; indeed, the latter can be obtained from the former by attaching a collar to that part of $Y_t \subset \partial(Y \cap S^3_{(-\infty, t]})$ which lies over the complement of $\pi(h_i)$ in $\pi(Y_t)$. Hence, in particular, the outside handles alone describe a handlebody structure for Y, and we have

LEMMA 2.2. *For a compact critical level imbedding $g: S^3 \to S^4$, the sum of the number of inside (resp. outside) 1- and 3-handles is one more (resp. less) than the sum of the number of inside (resp. outside) 0- and 2-handles.*

Proof. The component Y is a homotopy 4-ball. Thus the sum of the number of 1- and 3-handles is one less than the number of 0- and 2-handles. This proves 2.2 for outside handles; for inside handles, first switch inside and outside, by applying 1.3 to the first 0 and last 3-handle. □

Note that the first 0-handle is always outside.

PROPOSITION 2.3. *If $g: S^3 \to S^4$ is a rectified compact imbedding such that all 0- and 1-handles (except possibly the first 0-handle) are on the inside or all on the outside then g is isotopic to the standard imbedding.*

Proof. The operation of 1.3 on the first 0-handle and last 3-handle shows that the cases are symmetric. If the 2-handles also are on the same side (say the inside) then Y has no

1- or 2-handles, so it is a standard PL 4-ball in S^4. Then $\partial Y = g(S^3)$ is isotopic to the standard S^3 in S^4.

If not all the 2-handles are on the same side (say the outside) as the 0- and 1-handles, let h_i be the first 2-handle to lie on the inside. Since each handle below h_i is attached on the outside, $W_{t_i - 1}$ actually can be isotoped to lie entirely in $S^3_{t_i - 1}$.

For c_i, the core disk of h_i, $S^3_{[t_i - 1, t_i]} \cap \pi^{-1}(\pi(c_i))$ is a PL 3-ball D in Y. The closure of each component of $Y - D$ has boundary a PL 3-sphere with a rectified imbedding in which all the 0- and 1-handles are outside and there is at least one fewer 2-handle than for $g(S^3)$. By induction, each component is a PL 4-ball. Then Y is the boundary connected sum of two PL 4-balls along a 3-ball, so Y is a PL 4-ball. Then $g(S^3) = \partial Y$ is standard. □

COROLLARY 2.4. *Any genus one critical level imbedding $g: S^3 \to S^4$ is isotopic to the standard imbedding.*

Proof. By definition, the associated handlebody structure for $g(S^3)$ has as many 0-handles as 1-handles. Since $\chi(S^3) = 0$ there are as many 3-handles as 2-handles. Apply 1.5 to the 0- and 1-handles and, dually, to the 2- and 3-handles, obtaining an embedding with a single handle of each index. Then, after possibly applying 1.3, g is compact, and has its 0- and 1-handle on the same side. The proof then follows from 2.3. □

§3. THE GENUS TWO CASE

According to 1.5, a genus two critical level imbedding $g: S^3 \to S^4$ is isotopic to one with at most two 0-handles and three 1-handles. Also, since $\chi(S^3) = 0$, there are k 3-handles and $k + 1$ 2-handles. Then 1.5 applied to the dual imbedding isotopes g so that $g(S^3)$ also has at most two 3-handles and three 2-handles.

PROPOSITION 3.1. *A genus two critical level imbedding may be isotoped so that $g(S^3)$ has a single* 0*-handle, two* 1*-handles, two* 2*-handles, and a* 3*-handle.*

Proof. From the remarks above, there could be at most one more handle of each index. Suppose g is rectified and compact and there are two 0-handles h_1, h_2, and three 1-handles h_3, h_4, h_5. If either h_3 or h_4 is a connecting 1-handle it can be cancelled by 1.4. Hence h_5 is a connecting 1-handle. With no loss of generality suppose h_2 is an outside 0-handle. If both h_3 and h_4 are attached to the same 0-handle, h_5 may be cancelled by 1.4, so, with no loss of generality assume h_3 is attached to h_1 and h_4 to h_2. In order that Y be connected (see 2.1) h_5 is an outside 1-handle. If h_4 is also outside then, by 1.1, h_4 and h_5 can be interchanged and h_5 cancelled via 1.4. If h_4 is inside and h_3 is outside then, by 1.1, h_3 and h_4 can be interchanged, followed by h_3 and h_5 and again h_5 can be cancelled. Thus h_3 and h_4 must both be inside. Then, by 2.2, either all 2-handles are inside or there is a single outside 2-handle and an outside 3-handle. In the former case 2.1 implies that Y is a PL 4-ball with two 0-handles, a single 1-handle and no 2-handles, hence is standard. Then $\partial Y = g(S^3)$ is also standard. In the second case an argument dual to that above shows that the single outside 2-handle must be the first 2-handle h_6. This case requires the following extensive analysis.

Set $t_i = i - 1/2$, $i = 1, \dots, 6$, and let $W = \pi(X_5) \subset S^3$, T_1 and T_2 be the tori of $\pi(M_4)$ which bound $\pi(h_1) - \pi(h_3)$ and $\pi(h_2) - \pi(h_4)$ respectively, D_1 and D_2 the disks $\pi(c_3')$ and $\pi(c_4')$ respectively (so ∂D_i is a longitude of T_i, $i = 1, 2$), let E be the disk $\pi(c_6) \subset W$ and parameterize $\pi(h_5)$ by $I \times B^2$ so that $\pi(c_5) = I \times \{0\}$, $\{0\} \times B^2 \subset T_1$ and $\{1\} \times B^2 \subset T_2$. Isotope all these to general position so that in particular $I \times B^2$ intersects $D_1 \cup D_2$ in a collection $\{p_i\} \times B^2$ of disks parallel in $I \times B^2$.

Claim. There are disks D_1' and D_2' in W with $\partial D_i' = \partial D_i$, $i = 1, 2$, such that $(I \times B^2) \cap (D_1' \cup D_2') = 0$.

Proof of Claim 1. This is a standard innermost disk, outermost arc argument on $E \cap (D_1 \cup D_2)$.

First observe that it suffices to find disks disjoint from $I \times B^2$ and each other whose boundaries are isotopic to longitudes of T_1 and T_2, since any two longitudes are isotopic in a punctured torus (the punctures are $\partial I \times B^2$). Choose new disjoint disks D_1 and D_2 to minimize first the number of (transverse) intersections with $I \times \{0\} \subset I \times B^2$ and then to minimize the number of components of intersection of $E \cap (D_1 \cup D_2)$.

Suppose, in fact, that $E \cap (D_1 \cup D_2) = \phi$. Then either $I \times B^2$ is disjoint from the D_i and we are done or ∂E can be isotoped off $I \times \partial B^2$ in ∂W. In the latter case, since ∂E separates, it must be parallel in (say) T_1 to $\{0\} \times \partial B^2$. Then D_1 and $I \times \partial B^2$ lie in different components of W–E and so are disjoint. The component of W–E containing $I \times \partial B^2$ is then *PL* equivelent to a solid torus; a longitude can then be found on its boundary disjoint from the collared 2-cell $E \cup (I \times B^2)$ and again we are done.

On the other hand, suppose $E \cap (D_1 \cup D_2) \neq \Phi$. By replacing a disk in the D_i with a disk in E we could eliminate a circle of intersection in $E \cap (D_1 \cup D_2)$ if any exists. Thus $E \cap (D_1 \cup D_2)$ consists entirely of arcs. Let C be a cell in E–D with $\partial C = \alpha \cup \beta$, α a subarc of ∂E and β an arc in (say) $E \cap D_1$. There are four possibilities.

(i) The ends of β both lie on ∂D_1

(ii) One end of β lies on ∂D_1 and the other on the boundary of a disk component of $(I \times B^2) \cap D_1$.

(iii) The ends of β lie on boundaries of distinct components of $(I \times B^2) \cap D_1$.

(iv) The ends of β both lie on the boundary of the same disk component $\{p\} \times B^2$ of $(I \times B^2) \cap D_1$.

In case (i), either $(I \times B^2) \cap (D_1 \cup D_2) = \phi$ and we are done, or $(I \times B^2) - (D_1 \cup D_2)$ consists of two or more components and α intersects only that containing $\{0\} \times B^2$. In particular, α is isotopic rel end points in $\partial W - (D_1 \cup D_2)$ to an arc lying entirely in $T_1 - (\{0\} \times B^2)$. A regular neighborhood of $D_1 \cup C$ then has boundary consisting of three disks, one parallel to D_1, the other two, D and D', obtained by alternately replacing each disk component of $D_1 - B^2$ by C. The disks D and D' cannot both have innessential boundary in ∂T_1 or D_1 would also. Hence one of them has boundary a longitude of T_1; it also intersects $I \times B^2$ in no more components than does D_1 and intersects E in a least one fewer, a contradiction.

In cases (ii) and (iii), C may be used to isotope D_1 so it intersects $I \times B^2$, in one or two fewer disks respectively.

In case (iv), α cannot be isotoped in ∂W to an arc in $\{p\} \times \partial B^2$, for this transforms β into a circle of intersection of E and D_1, which can be eliminated as above. It follows that α can be isotoped to an arc which travels along a component $J \times \partial B^2$ of $(I \times \partial B^2) - (D_1 \cup D_2)$ (from $\{p\} \times \partial B^2$ to either $\{0\} \times \partial B^2$ or $\{1\} \times \partial B^2$) then around a longitude of T_1 or T_2, then returns along $J \times \partial B^2$ to $\{p\} \times \partial B^2$. Denote by T_i the torus which α intersects. The union of $J \times B^2$ and a regular neighborhood of C has boundary consisting of two cylinders F and F' each with one end in D_1 and the other end a longitude of T_i. Let D and D' be the disks in D_1 bounded by an end of F and F' respectively, with $D \subset D'$. Note that $\{p\} \times B^2 \subset D' - D$. Then $F \cup D$ is a disk with boundary a longitude of T_i and having fewer intersections with $I \times \{0\}$ than has D_1. Hence, $i = 2$, and $\#((I \times \{0\}) \cap D) \geq \#((I \times \{0\}) \cap D_2$. But then the disk $(D_1 - D') \cup F' \cup D_2$ intersects $I \times \{0\}$ in one less point than does D_1, producing a contradiction and proving the claim.

Claim 2. There is a homeomorphism $\phi : S^3 \to S^3$ such that $\phi | \pi(Y_4)$ is the identity and $\phi(D_2') = D_2$.

Proof. $\pi(Y_4)$ consists of two components, L_1 and L_2 bounded by T_1 and T_2 respectively. Since $S^3 - L_2 \simeq S^1 \times D^2$ there is an isotopy $\phi_t: S^3 \to S^3$, fixing L_2, from the identity to a homeomorphism that carries D_2' to D_2. Since $D_2' \cap L_1 = \phi$, $\phi_1(L_1) \subset S^3 - (L_2 \cup D_2)$. Now $L_2 \cup D_2$ has regular neighborhood N a 3-ball, so exploiting a collar in S^3 between ∂N and the boundary of a 3-ball well within L_2, there is also an isotopy $\phi_t': S^3 \to S^3$, fixed on N, from the identity to a homeomorphism such that $\phi_1'|L_1 = \phi_1|L_1$. Then $\phi_1'^{-1}\phi_1: S^3 \to S^3$ is the required homeomorphism, proving Claim 2.

Proof of 3.1 (*completion*). Let $HY \to S^3 \times \mathbb{R}$ be the PL imbedding defined as the inclusion on $Y \cap S^3 \times (-\infty, 4]$ and the inclusion composed with $\phi \times \mathrm{id}$ on $Y \cap S^3 \times [4, \infty)$. Then $H|\partial Y$ is a critical level imbedding, with handles h_1, h_2, h_3 and h_4 those of $g(S^3)$ but the fifth handle is $\phi(h_5)$. Since $\pi(h_5) \cap D_2' = \phi$, $\pi\phi(h_5) \cap \pi(h_4) = \phi$ and h and $\phi(h_5)$ can be interchanged. As above, this means $H(Y)$ is a standard PL 4-ball. Then so is Y, so $g(S^3) = \partial Y$ is standard. □

In order to complete the proof of the genus two case, the two following lemmas are needed. The first is apparently due to Tsukui [6]; a much broader statement was proven using different techniques in [4]. The second is really a corollary of the central theorem of [4].

LEMMA 3.2. *Suppose a 1-handle* $(I \times D^2, \partial I \times D^2) \subset (S^1 \times D^2, S^1 \times \partial D^2)$ *is removed from* $S^1 \times D^2$ *producing a 3-manifold* $W \subset S^1 \times D^2$ *with* ∂W *a genus two surface. If* ∂W *is compressible in* W, *then there is a properly imbedded disk in* W *whose boundary lies in* $(S^1 \times \partial D^2) \cap \partial W$ *and which is properly isotopic in* $S^1 \times \partial D^2$ *to a meridional disk* (*point*) $\times D^2$.

Proof. See [6, 3.6] or [4]. □

Define a knot γ in a compact 3-manifold M to have *tunnel number one* if one can attach some 1-cell in M to γ so that the regular neighborhood of the resulting complex has complement a solid handlebody. The following was shown in [4]. Suppose γ is a tunnel number one knot in S^3 with regular neighborhood $\gamma \times D^2$. If there is a planar surface $P \subset S^3 - (\gamma \times \mathring{D}^2)$ such that ∂P is $(2k+1)$ longitudes of $\gamma \times \partial D^2$, then γ is trivial. In particular, 0-framed surgery on a tunnel number one knot γ yields $S^2 \times S^1$ only if γ is the unknot.

LEMMA 3.3. *Let* W *be the 4-manifold obtained from* $S^1 \times D^3$ *by attaching a 2-handle to the tubular neighborhood* $\gamma \times D^2$ *of a curve* γ *in* $\partial(S^1 \times D^3) = S^1 \times S^2$ *such that*

(a) $\partial W \simeq S^2$

(b) γ *has tunnel number one in* $S^1 \times S^2$.

Then W *is PL homeomorphic to* D^4.

Proof. The boundary of the cocore of the 2-handle is a circle $\bar{\gamma}$ in $S^3 = \partial W$. The complement of its tubular neighborhood in S^3 is just the complement of $\gamma \times D^2$ in $S^1 \times S^2$, so $\bar{\gamma} \subset S^3$ has tunnel number one. Furthermore, 0-framed surgery on $\bar{\gamma} \subset S^3$, produces $S^1 \times S^2$, by construction. Hence $\bar{\gamma}$ is trivial in S^3[4]. Thus there is a 2-disk in $\partial W \simeq S^2$ whose boundary is $\bar{\gamma}$. This can be isotoped to a disk in $S^1 \times S^2 - (\gamma \times \mathring{D}^2)$ whose boundary is a meridian of $\gamma \times \partial D^2$. The union of the disk and a meridional disk of $\gamma \times D^2$ is a 0-sphere intersecting γ in precisely one point and bounding a copy of D^3 in $S^1 \times D^3$. The results follows by standard handle cancellation. □

THEOREM 3.4. *A genus two critical level imbedding* $g: S^3 \to S^4$ *is isotopic to the standard imbedding.*

Proof. The previous discussion has shown the g can be simplified so that it has one 0-handle h_1 followed by two 1-handles h_2 and h_3, two 2-handles h_4 and h_5 and a 3-handle. We can also assume that $t_i = i - 1/2$, $i = 1, \ldots, 6$, h_2 is an inside 1-handle, and h_3 is an outside 1-handle.

Case (i). h_4 is an inside 2-handle. The h_5 is an outside 2-handle. This case is symmetric, meaning that dual imbedding also has the 1- and 2-handles appearing in order: inside, outside, inside, outside.

First note that $\pi_1(Y_3) \simeq \pi_1(Y_2)*Z$, since Y_3 is obtained from Y_2 by attaching a 1-handle. In general, if C is a component of the complement of a genus two handlebody in S^3, either $\pi_1(C) \simeq Z * Z$ or there is at most one isotopy class of non-separating disks in C[6]. However, the projections of the cocore of h_3 and the core of h_4 in S^3 cannot be properly isotopic in $\pi(Y_3)$, for otherwise their union along the collar between Y_2 and Y_3 would be a non-separating 2-sphere in $g(S^3)$. Therefore $\pi_1(Y_2) \simeq \mathbb{Z}$ and M_2 is an unknotted torus in S^3_2.

Dually, M_4 is an unknotted torus in S^3_4.

Examine the 4-manifold $V = Y \cap S^3_{[0,4]}$. The only outside handles added to Y through level 4 have been h_1 and h_3, so $V \simeq S^1 \times D^3$. Furthermore, a collared handle description of $\partial Y - Y_4$ is given by the handles h_1, h_2, h_3 and the collars between their levels. Since M_4 is unknotted, Y_4 is a solid torus; then M_3 provides a Heegaard splitting of ∂V and Y_4 is a tunnel number one knot in ∂V. Now Y itself is obtained by attaching the 2-handle h_5 to V via a longitude of Y_4. Since $\partial Y \simeq S^3$, 3.3 applies, and Y is a PL 4-ball.

Case (ii). h_4 is an outside 1-handle.

$\pi(X_2)$ is a solid torus $S^1 \times D^2$ from which the 1-handle $\pi(h_3)$ is removed to obtain $\pi(X_3)$. Furthermore, $\pi(\partial X_3)$ is compressible in $\pi(X_3)$, for the core of $\pi(h_4)$ is a compressing disk. By 3.2 there is a non-separating disk D in $\pi(X_3)$ such that ∂D lies in $\pi(\partial X_2) \cap \pi(\partial X_3)$ and is isotopic in $\pi(\partial X_2)$ to a meridian of $\pi(X_2)$, i.e. to the projection of the cocore of h_2. Apply 1.1 to interchange h_2 and h_3, then apply 1.3 to switch inside and outside. The 1- and 2-handles then appear in order: inside, outside, inside, outside as in case 1. □

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