1990 PROCEEDINGS OF KAIST MATHEMATICS WORKSHOP

LECTURES ON THE THEORY OF SUTURED 3-MANIFOLDS

MARTIN G. SCHARLEMANN[1]

Department of Mathematics
University of California
Santa Barbara, CA 93106, U.S.A

This manuscript is meant to accompany a series of lectures given at the Korea Advanced Institute of Science and Technology in August, 1990. The topic is the theory of sutured 3-manifolds, and the charge was to present three lectures, from the basic material to the recent developments in the area.

I've chosen here just those parts of basic 3-manifold theory necessary to make sense of the development of sutured manifold theory, and merely outlined a few of the proofs whose understanding is crucial to this development. Complete proofs of these and other classical theorems of 3-manifold theory can be found in the excellent texts of Hempel [**He**] and Jaco [**J**].

The origin of sutured 3-manifold theory can be traced back to Thurston's discovery of a norm on the homology of 3-manifolds [**Th**]. David Gabai combined this discovery with Waldhausen's notion of a hierarchy of a 3-manifold, producing the seminal work [**Ga**]. Many applications of sutured manifold theory have been found; a partial list is given in the references.

The viewpoint of these notes in the "defoliated" viewpoint presented in [**Sc**], which is a good source for detailed proofs.

1. 3-manifolds

DEFINITION: A 3-manifold M is a separable metric space in which every point has a neighborhood homeomorphic to an open set in $R^3_+ = R^2 \times [0, \infty)$. A point in M which has no neighborhood homeomorphic to R^3 is called a boundary point of M. The set of all boundary points is a surface (2-manifold) denoted ∂M.

[1]Supported in part by a National Science Foundation grant.

1.1. THEOREM (Moise, Bing). *Any compact 3-manifold is homeomorphic to a finite simplicial complex, unique up to homeomorphism and common subdivision.*

1.2. Suppose M is a compact 3-manifold and K is a finite simplicial complex to which M is homeomorphic. Then K suggests a natural decomposition of M, called a *handlebody* decomposition, or a ball-rod-plate decomposition (See Figure 1):

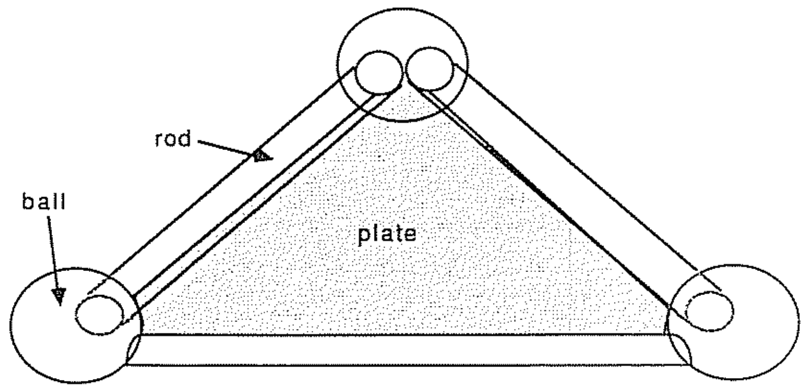

Figure 1

The set of *balls* (or 0-*handles*), ${}^0M$, is a regular neighborhood in M of the set of vertices of K. So ${}^0M$ is a finite union of 3-balls in M.

We can assume that each edge in K restricts in $M - {}^0M$ to an arc whose ends lie in $\partial({}^0M)$. This arc has a regular neighborhood homeomorphic to $I \times D^2$, with $\partial I \times D^2 \subset \partial({}^0M)$. This regular neighborhood is called a *rod* (or 1-*handle*). Let ${}^1M$ denote the union of ${}^0M$ with a 1-handle at each edge.

By choosing ${}^1M$ close enough to the 1-skeleton of K we can assume that each 2-simplex in K restricts in $M - {}^1M$ to a disk D^2. Such a disk has a regular neighborhood homeomorphic to $D^2 \times I$, with $\partial D^2 \times I \subset \partial({}^1M)$. This regular neighborhood is called a *plate* (or 2-*handle*), and again denote by ${}^2M$ the union of ${}^1M$ with a 2-handle at each 2-simplex.

Finally, $M - {}^2M$ is a collection of 3-balls, called 3-*handles*, one in each 3-simplex.

1.3. An orientation on a simplex is a choice of ordering of its vertices, up to even permutation. A vertex v_0 of an s-simplex σ and an orientation

$[\tau] = [v_1, \ldots, v_s]$ of its opposite face τ in σ determine an orientation on σ by just appending: $[\sigma] = [v_0 v_1, \ldots, v_s]$. In this case we say the orientation $[\tau]$ is inherited from $[\sigma]$. If τ is a 2-simplex of a triangulated 3-manifold M, either τ lies in ∂M or τ is the face of exactly two 3-simplices. If orientations on the 3-simplices can be chosen so that the induced orientation on each interior 2-simplex is never consistent, we say that M is *orientable* and the choice is an *orientation*.

Henceforth we'll consider only orientable compact 3-manifolds, though this restriction isn't always necessary. This is equivalent to requiring that ^1M be a *solid handlebody*, that is, ^1M is homeomorphic to the regular neighborhood of a 1-complex (graph) in R^3. (See Figure 2)

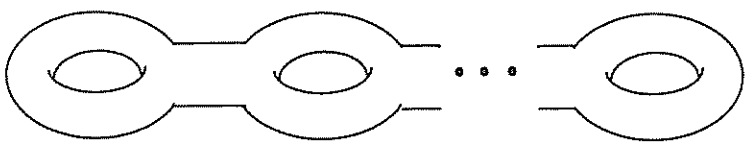

Figure 2

1.4. It's easy to see that for any compact orientable 3-manifold M, $M - {}^1M$ is also a solid genus g handlebody. This gives the remarkable description:

THEOREM. *Any compact orientable 3-manifold is the union of two solid handlebodies along their boundary.*

This description of M is called a *Heegaard splitting* of M and has motivated quite a bit of research on 3-manifolds. But it's not used here, and we won't pursue it.

2. Submanifolds and general position

2.1. A *submanifold* S of a 3-manifold M will always mean a proper PL submanifold. PL means there's a simplicial subdivision of M so that S is a subcomplex. *Proper* means that $\partial M \cap S = \partial S$ and for any compact $K \subset M$, $K \cap S$ is also compact. If S is a 1-manifold then a regular neighborhood $\eta(S)$ of S is homeomorphic to $S \times D^2$. If S is a 2-manifold, $\eta(S)$ is homeomorphic to $S \times I$, and we'll denote the image of $S \times \partial I$ by $\partial\eta(S)$. Each simplex τ in S is the face of two different 3-simplices of M. Suppose orientations are given on S and M. The 3-simplex of M whose orientation is consistent with the

orientation on τ we'll say is on the *positive* side of S. The copy of S in $\partial\eta(S)$ lying on the positive [resp. negative] side will be denoted S_+ [resp. S_-]. A good way to picture the orientation is by the induced normal orientation: pick the normal direction to S which points into the negative side from the positive side.

Any two submanifolds of a 3-manifold M can be jiggled slightly to intersect in a particularly nice say. Any pair of lines in 3-space may be made disjoint by moving them very slightly. Similarly a line and a plane may be made to intersect in at most a point by moving slightly, and two planes may be made to intersect in a line. The situation is similar for submanifolds of 3-manifolds: If S, $T \subset M$ are submanifolds of dimension k and ℓ respectively, then after a slight movement of S or T we may take $S \cap T$ to be a manifold of dimension $k + \ell - 3$. In other words, if S and T are 1-manifolds, they can be made disjoint. If $\dim(S) = 1$ and $\dim(T) = 2$, then they can be made to intersect in a discrete set of points. If $\dim(S) = \dim(T) = 2$, then $S \cap T$ can be made to be a 1-manifold, i.e., a union of arcs and circles. In these cases, S and T are said to be in *general position*. The number of components of $S \cap T$ is denoted $|S \cap T|$.

2.2. A circle or proper arc γ in a surface S is called *inessential* if $S - \gamma$ has two components, at least one of which is a disk. Otherwise γ is called *essential*. If Γ is a proper compact 1-submanifold in a surface S, and some circle component of Γ is inessential, then there is a component bounding a disk in S which is disjoint from Γ. This component is called an *innermost circle* of Γ and the disk it bounds is an *innermost disk*. Similarly, if there's an inessential arc in Γ then there's one for which the disk it cuts off from S is disjoint from Γ. This is called an *outermost arc* of Γ and the disk it cuts off is called an *outermost sector*. (See Figure 3).

2.3. Suppose S and T are a pair of surfaces in general position in a compact orientable 3-manifold M, and $S \cap T$ contains a closed component which is inessential in S. Then it contains an innermost inessential circle cutting off a disk D in S which is disjoint from T. Let $\eta(D) \cong D \times I$ be a relative regular neighborhood of D, so that $\partial D \times I \subset T$. Then the surface $T' = [T - (\partial D \times I)] \cup (D \times \partial I)$ is said to be obtained from T by *2-surgery along* D. See Figure 4. Note that the Euler characteristic $\chi(T') = \chi(T) + 2$, since an annulus ($\chi = 0$) had been replaced by two disks (each with $\chi = 1$).

Moreover $|S \cap T'| < |S \cap T|$. If ∂D is essential in T then T' contains no new 2-spheres.

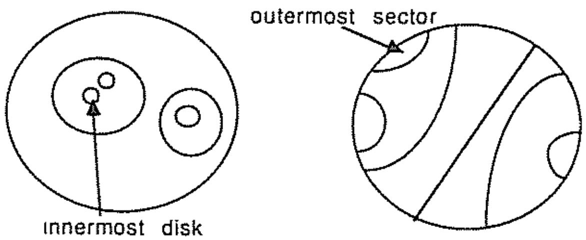

Figure 3

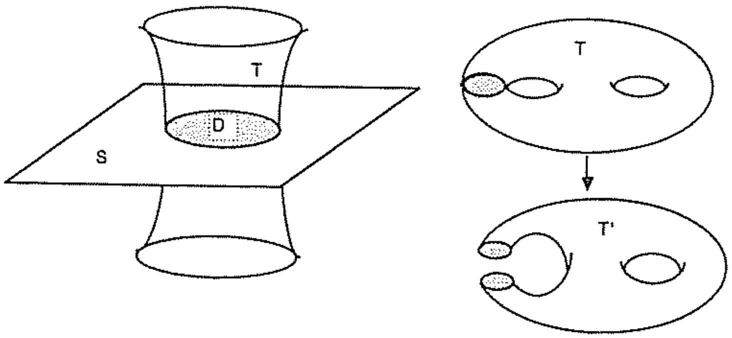

Figure 4

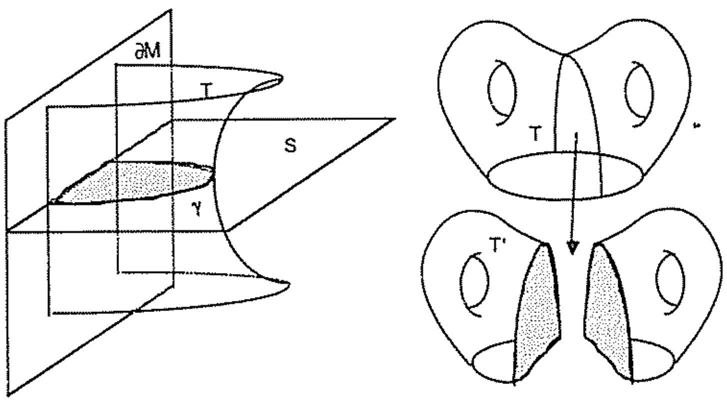

Figure 5

A similar construction can be made on an inessential outermost arc γ of $S \cap T$ in T. The new manifold T' has $|S \cap T'| < |S \cap T|$ and $\chi(T') = \chi(T)+1$, but $\partial T' \neq \partial T$. If γ is essential in T, T' contains no new 2-disks. See Figure 5.

3. Connected sums

Suppose M_1 and M_2 are two oriented 3-manifolds, and B_1, B_2 are two 3-balls, with $B_i \subset M_i$. Let M_1' and M_2' be the 3-manifolds obtained by removing the interiors of the B_i and $M_1 \# M_2$ the 3-manifold obtained from $M_1' \cup M_2'$ by identifying ∂B_1 and ∂B_2 via some orientation reversing homeomorphism. Any two orientation preserving imbeddings of the 3-ball in a 3-manifold are isotopic, and so are any two orientation reversing homeomorphisms of the 2-sphere, so this operation is well-defined. The orientations of M_1 and M_2 determine an orientation on M. Note that always $M \cong M \# S^3$.

3.1. Definition: A 3-manifold M is *prime* if whenever $M = M_1 \# M_2$, one of the M_i is S^3. A non-prime 3-manifold is sometimes called a *composite manifold*. Suppose M' is obtained from M by removing the interior of k 3-balls. We say M' is a *punctured* M. Unless $M = S^3$ and $k = 1$, M' is never prime. Indeed, $M' = M \# k(B^3)$.

Example: It is a theorem of Alexander that any imbedding of S^2 in S^3 divides S^3 into two 3-balls. Hence S^3 is prime.

Example: $S^2 \times S^1$ contains an imbedded 2-sphere $S^2 \times \{\text{point}\}$ not bounding any 3-ball, yet it's prime. But this is the *only* example of a prime orientable 3-manifold containing a 2-sphere not bounding a ball. Indeed, consider any sphere S in a connected orientable 3-manifold M.

If $M - \eta(S)$ has two components M_1' and M_2', then $M = M_1 \# M_2$, where M_i is obtained from M_i' by attaching a 3-ball. If neither M_i' is a 3-ball, then M is not prime.

If $M' = M - \eta(S)$ is still connected, then let α be an arc in M' from one side of $\eta(S)$ to the other. Then $\eta(S) \cup \eta(\alpha)$ is just the manifold obtained from $S^2 \times S^1$ by removing a 3-ball. Let $M_1' = M' - \eta(\alpha)$, and M_1 be the manifold obtained by attaching a 3-ball to $\partial M_1'$. Then $M = M_1' \# S^2 \times S^1$. Hence if M is prime, M_1' is a 3-ball and $M = S^2 \times S^1$.

We say that a 3-manifold is *irreducible* if any S^2 in it bounds a 3-ball. The following theorem shows that it's often reasonable to restrict interest only to irreducible 3-manifolds.

3.2. THEOREM. *Any compact connected orientable 3-manifold can be written in a unique way as the connected sum of a finite number of prime 3-manifolds.*

PROOF: The proof consists of two parts.

Existence of a prime decomposition (Kneser): The idea, due to Kneser, is that there's only room for a finite number of imbedded 2-spheres in a given compact 3-manifold before some pair of 2-spheres is parallel (i.e. the region between them is the product $S^2 \times I$). To see this, choose a fixed triangulation of M and consider the associated ball-rod-plate decomposition of M. A finite collection S of disjoint 2-spheres in M can be put in general position with respect to this structure. In this context, this means that

a) S is disjoint from the 3-handles (so it lies in ${}^2M$)
b) S intersects each plate $D^2 \times I$ in a finite number of disks parallel to $D^2 \times \{0\}$
c) S intersects each rod $D^2 \times I$ in $\Gamma \times I$, where Γ is a finite collection of arcs in D^2
d) S intersects the boundary of each 0-handle in a set of simple closed curves.

The proof proceeds in two steps: First isotope S to minimize the size of each collection in b)–d). Then observe that in each 0, 1, or 2-handle H most components of $S \cap H$ must be parallel in H. (It's helpful here to first reduce to the case in which $S \cap {}^0M$ is a set of disks.) Indeed the number of possible pairs of adjacent components of $S \cap H$ that are not parallel in H has a maximum determined only by the triangulation, not by S. If S is large enough, at least one adjacent pair of spheres in S can be found that contains none of the bad components in any of the handles, so these will be parallel in M.

Uniqueness of the prime decomposition (Milnor): We want to show that if $M = M_1 \# M_2 \# \cdots \# M_k \cong N_1 \# N_2 \# \cdots \# N_\ell$, where each M_i and each N_j is prime, then $k = \ell$ and, after a permutation, $M_i \cong N_i$. To simplify things a bit, we'll assume that no N_j is $S^2 \times S^1$, so each N_j is irreducible. Let S be a sphere in M chosen so that one component of $M - S$ is a punctured M_k. If S is disjoint from some collection T of spheres decomposing $N_1 \# N_2 \# \cdots \# N_\ell$, then, since any sphere in N_j bounds a ball,

it would follow that each component of $M - S$ is a punctured sum of N_j's. Since M_k is prime, in fact M_k would then be homeomorphic to one of the N_j's, completing an inductive step.

Thus it suffices to show that S is disjoint from some collection T of spheres decomposing $N_1 \# N_2 \# \cdots \# N_\ell$. So assume T has been chosen to minimize $|S \cap T|$. If $|S \cap T| \neq 0$, let D be an innermost disk cut off by T in S. ∂D divides some component T_i of T into two disks E and E', each disjoint from D. The component N_j' of $M - T$ in which D lies is a punctured N_j. Since N_j is prime, one of the spheres obtained by 2-surgery on T_i along D bounds a possibly punctured sphere in N_j'. Replace T_i with the other sphere. See Figure 6. This reduces $|S \cap T|$.

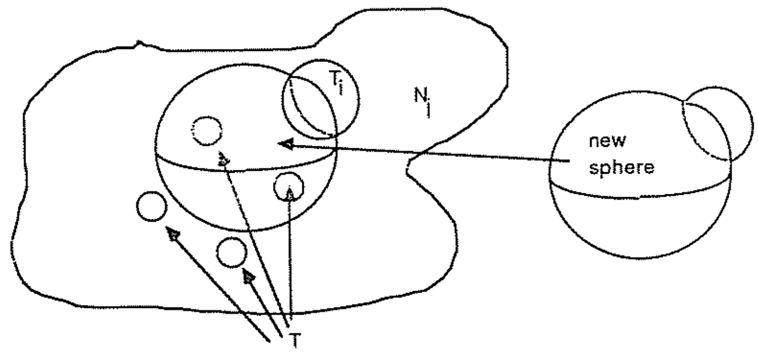

Figure 6

REMARKS: It's not true that an automorphism of a composite 3-manifold is necessarily isotopic to an automorphism that preserves the connected sum decomposition.

The following major theorem of Papakyriakopolous gives a simple criterion to detect reducibility:

SPHERE THEOREM. *If M is irreducible then $\pi_2(M)$ is trivial.*

4. Incompressible surfaces and hierarchies

As before, M is a compact orientable 3-manifold.

4.1. DEFINITION: A surface $(S, \partial S) \subset (M, \partial M)$ is *compressible* if either

a) S is a sphere bounding a ball in M or

b) There's a disk D in M so that $\partial D = D \cap S$ is an essential simple closed curve in S.

Otherwise S is *incompressible*.

4.2. DEFINITION: S is *∂-compressible* if either

a) S is a disk and $\partial S \subset \partial M$ bounds a disk D in ∂M such that $S \cup D$ bounds a ball in M or

b) There's a disk D in M so that $D \cap S = \partial D \cap S$ is a single essential arc in S and $\partial D - S$ is an arc in ∂M.

Otherwise S is *∂-incompressible*.

If ∂M is compressible we say that M is *∂-reducible*.

2.3 explains why we are interested in compressibility. If S satisfies 4.1b) or 4.2b) then it may be altered by a 2-surgery into a simpler surface. (See Figures 4 and 5).

4.3. LOOP THEOREM (Papakyriakopolous). *If S is incompressible then $\pi_1(S) \to \pi_1(M)$ is injective.*

4.4. A *non-separating* surface $(S, \partial S) \subset (M, \partial M)$ is a surface such that $M - S$ has no more components than M.

THEOREM. *If a compact orientable 3-manifold M contains a non-separating surface, then it contains an incompressible, ∂-incompressible surface.*

PROOF: Pick a non-separating surface of highest possible Euler characteristic χ. Any compression or ∂-compression would give a non-separating surface of higher χ. ∎

4.5. It requires a bit of algebraic topology to show that following:

THEOREM. *If a compact orientable 3-manifold M contains no non-separating surface, then ∂M is a union of 2-spheres.*

In particular, if M is prime, either M contains a non-separating surface or each component of M is either closed or a 3-ball.

4.6. DEFINITION: An irreducible compact orientable manifold M containing an incompressible surface is called a *Haken* manifold.

4.7. Suppose $S \subset M$ is an incompressible surface. Let M' be the manifold obtained from M by removing an open regular neighborhood of S. We say M' is obtained by *decomposing* M along S. We write $M \xrightarrow{S} M'$.

PROPOSITION. *M' is reducible if and only if M is reducible.*

PROOF: Suppose T is a reducing sphere for $M' \subset M$. If T is not a reducing sphere for M then it bounds a 3-ball in M, but not in M', so S must lie inside the 3-ball. But any surface in a 3-ball is compressible by Van Kampen's theorem and 4.3.

On the other hand, suppose T is a reducing sphere for M, chosen to minimize $|S \cap T|$. If $|S \cap T| = 0$ then T lies in M', so M' is reducible. Otherwise, consider an innermost disk D cut off by $S \cap T$ in T. Since S is incompressible, ∂D bounds a disk E in S. If $D \cup E$ bounds a ball in M', then E could be isotoped just beyond D to reduce $|S \cap T|$. Otherwise $D \cup E$ is a reducing sphere for M_1. ∎

4.8. DEFINITION: A *partial hierarchy* for M is a series of decompositions $M \xrightarrow{S_1} M_1 \xrightarrow{S_2} \cdots \xrightarrow{S_i} M_i \xrightarrow{S_{i+1}} \cdots$ such that each S_i is incompressible in M.

4.9. From 4.4 and 4.5 we see that any partial hierarchy can be extended so long as ∂M_n is not a union of 2-spheres. Moreover, to check whether M is irreducible, it suffices to check any 3-manifold in the decomposition. Much deeper is the following

THEOREM. *If each S_i is chosen also to be ∂-incompressible, then any partial hierarchy of M has finite length.*

The proof is analogous to that of 3.2. Instead of a bound on the number of spheres in M, there's a bound on the number of incompressible, ∂-incompressible surfaces in M.

When we can extend a partial hierarchy no further, it follows from 4.5 that ∂M_n is a union of spheres. In this case we say $M \xrightarrow{S_1} M_1 \xrightarrow{S_2} \cdots \xrightarrow{S_n} M_n$ is a *hierarchy* for M. From 4.7 it follows that M is irreducible if and only if M_n consists of 3-balls.

The existence of hierarchies for 3-manifolds is a surprising analogue of a well-known process on surfaces. Figure 7 shows a "hierarchy" for a genus 2 surface. Hierarchies have been very useful in 3-manifold theory. Theorems about Haken manifolds can often be deduced by reconstructing M from 3-balls via its hierarchy. Sutured manifold theory is one example of this idea, but with attention paid to orientation during the hierarchy.

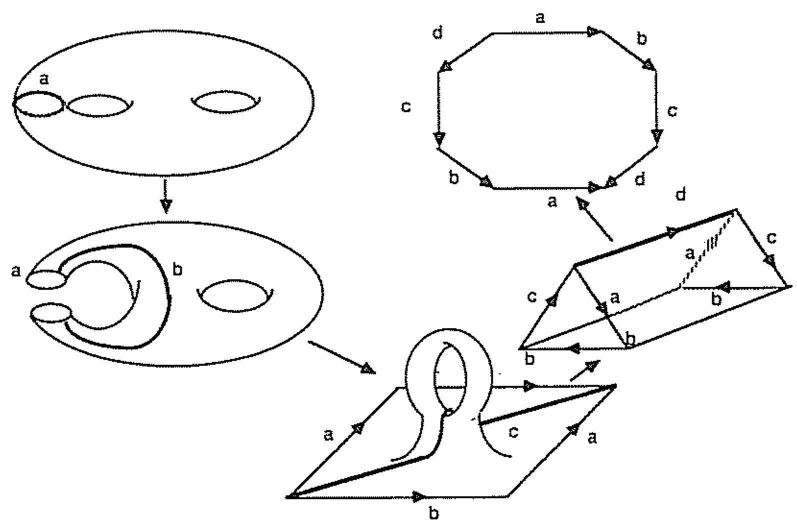

Figure 7

5. Homology in M and the Thurston norm

Henceforth we will assume that our manifolds are not only compact and orientable, but also irreducible and oriented. Any element in $H_2(M, \partial M)$ is the fundamental class of some imbedded non-separating surface $(S, \partial S) \subset (M, \partial M)$. The fundamental class of a connected imbedded surface in M is trivial in $H_2(M, \partial M)$ if and only if S is separating. So the homology of M carries very specific information about non-separating surfaces in M. This connection between the homology of M and surfaces in M was exploited by Thurston to put a (pseudo-)norm on the homology of any compact orientable irreducible 3-manifold [**Th**]. Here's a brief review:

5.1. DEFINITION: a) For S a connected orientable compact surface, define $\chi_-(S) = \max\{0, -\chi(S)\}$. That is, $\chi_-(S)$ is $-\chi(S)$ unless S is a sphere or a disk, in which case $\chi_-(S) = 0$.

b) For S not necessarily connected, define $\chi_-(S) = \sum\{\chi_-(S_i) \mid S_i$ a component of $S\}$. Equivalently, $-\chi_-(S)$ is the sum of the Euler characteristics of the non-simply connected components of S.

c) For N a subsurface of ∂M, and α in $H_2(M, N)$, define the *Thurston norm* of α to be

$$x(\alpha) = \min\{\chi_-(S) \mid (S, \partial S) \subset (M, N) \text{ with } [S, \partial S] = \alpha\}.$$

5.2. The Thurston norm has four important properties:

THEOREM.

i) *$x(\alpha)$ is always a non-negative integer.*
ii) *$x(p\alpha) = |p|x(\alpha)$ for any integer p.*
iii) *$x(\alpha + \beta) \leq x(\alpha) + x(\beta)$*
iv) *For any α, β there is an integer k so large that for all positive integers ℓ*

$$x(\alpha + (k + \ell)\beta) = x(\alpha + k\beta) + \ell x(\beta).$$

PROOF: i) is immediate since χ_- is never negative.

ii) is equally easy in one direction: Since $p\alpha$ can be represented by p parallel copies of any surface representing α, we have $x(p\alpha) \leq |p|x(\alpha)$.

For the other direction, let S be an oriented surface representing $p\alpha$, choose a base point x_0 in $M - S$, and define a function $f : M - S \to Z_p$ by setting $f(x)$ to be the intersection number of an arc from x_0 to x with S. Each $f^{-1}(i)$, i in Z_p, is a cobordism from one subsurface S_i of S to another, S_{i+1}. Hence for each i in Z_p, $p\alpha = [S] = \sum\{[S_j] \mid j \in Z_p\} = p[S_i]$, so $p(\alpha - [S_i]) = 0$. Since $H_2(M, N)$ has no torsion, $[S_i] = \alpha$. Then $\chi_-(S_i) \geq x(\alpha)$ so $x(p\alpha) = \sum\{\chi_-(S_i) \mid i \in Z_p\} \geq px(\alpha)$.

iii) The idea here is to take surfaces S and T so that $[S, \partial S] = \alpha$, $[T, \partial T] = \beta$, $\chi_-(S) = x(\alpha)$, $\chi_-(T) = x(\beta)$ and construct from them a surface U representing $\alpha + \beta$. If $\chi_-(U) \leq \chi_-(S) + \chi_-(T)$ we will be done. The construction of U is accomplished by taking the *double curve sum* of S and T, defined below. By doing 2-surgeries on outermost sectors and innermost disks of $S \cap T$ in S and T we may, without increasing their χ_-, alter them so that any component of $S \cap T$ is essential in both S and T. Then apply 5.4 below.

iv) This is a formal consequence of i)-iii). See [Sc, 1.5].

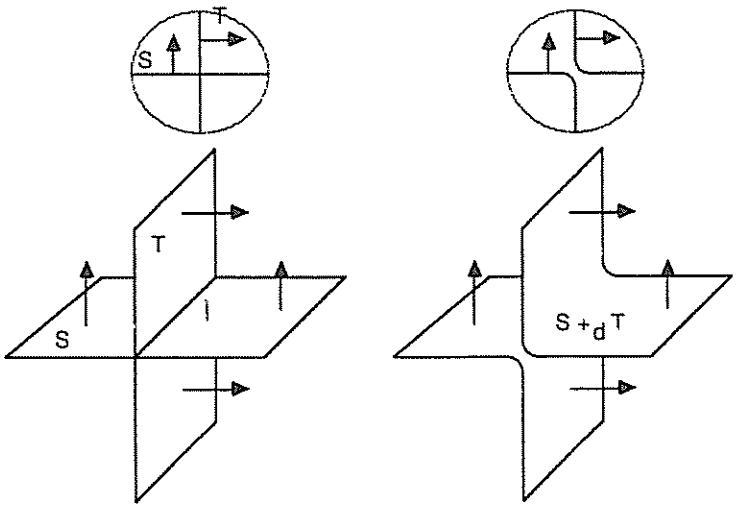

Figure 8

5.3. DEFINITION: Suppose S and T are oriented surfaces in general position in M. Then a neighborhood of $\Gamma = S \cap T$ in M intersects $S \cup T$ in the product of Γ and the left side of figure 8 (normal directions shown). The *double curve sum* $S +_d T$ of S and T is defined as the oriented surface obtained by replacing $S \cup T$ in this neighborhood with the product of Γ and the right side of figure 8.

5.4. LEMMA. *If every component of* $\Gamma = S \cap T$ *is essential in* S *and in* T*, then* $\chi_-(S) + \chi_-(T) = \chi_-(S +_d T)$.

PROOF: The construction of the double curve sum involves removing a subsurface of $S \cup T$ and then gluing it back in differently, so the operation preserves Euler characteristic. It suffices then to show that no new disks or spheres are created. Any disk or sphere in $S +_d T$ either lies entirely in S, entirely in T, or contains bits of both. In the latter case, the bits are glued together along components of Γ. An outermost arc or innermost circles of Γ would be inessential in either S or T.

5.5. DEFINITION: An oriented surface $(S, \partial S)$ is *taut* if,

a) S is incompressible and

b) S has minimal χ_- of all surfaces representing $[S, \partial S]$ in $H_2(M, \eta(\partial S))$. That is, for x the Thurston norm on $H_2(M, \eta(\partial S))$, we have $x([S]) = \chi_-(S)$.

Note that 5.5a) follows from 5.5b) unless S is a torus or an annulus.

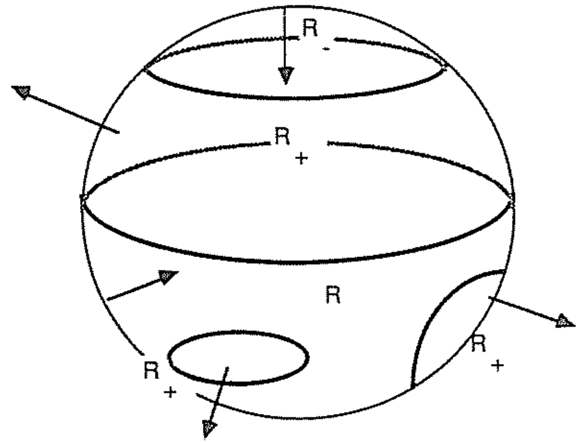

Figure 9

6. Sutured manifolds

6.1. DEFINITION: A sutured manifold (M, γ) is a compact oriented 3-manifold M together with a partition of $\partial M = R_+ \cup_\gamma R_-$ as the union of two surfaces R_+ and R_- along their common boundary, a collection of simple closed curves γ, called the *sutures*. R_+ is oriented so that its normal vector points outward and R_- so it points inward. See Figure 9.

REMARK: The orientations of $R_\pm$ induce a common orientation on γ.

Suppose $(S, \partial S) \subset (M, \partial M)$ is an oriented surface in a sutured manifold (M, γ). Then the orientation of S induces orientations on $S_\pm \subset \partial\eta(S)$. Let M' be the manifold obtained by decomposing M along S. Then any point in $\partial M'$ was either a point in ∂M or in $S_\pm$. Hence any point in $\partial M'$ which is neither in $\partial S_\pm$ nor in γ has a natural normal orientation. If the normal orientations on both sides of a point in $\partial S_\pm$ agree, use that normal orientation also at that point. Otherwise, regard the point as lying in a suture. In this manner M' becomes also a sutured manifold with a set of sutures γ'. We say that $(M,\gamma) \xrightarrow{S} (M',\gamma')$ is a *sutured manifold decomposition*. This operation is shown schematically one dimension lower in Figure 10, a decomposition of an annulus by an oriented arc.

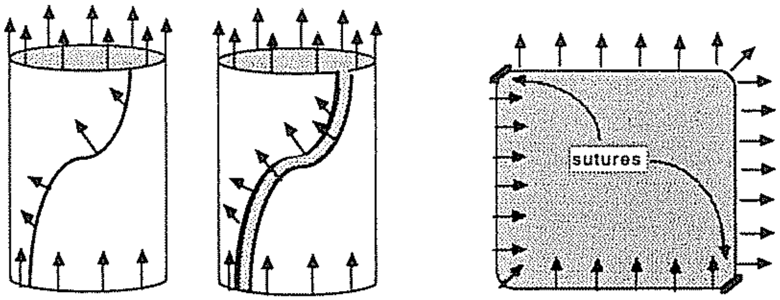

Figure 10

6.2. PROPOSITION. *Suppose $(M,\gamma) \xrightarrow{S} (M',\gamma')$ is a sutured manifold decomposition. Push $R_\pm$ slightly into M rel $\gamma = \partial R_\pm$. Then the subsurfaces $R'_\pm$ into which γ' divides $\partial M'$ are given by $R'_\pm = R_\pm +_d S$.*

PROOF: R'_+ is mostly the union of R_+ and S_+. To verify the proposition it suffices then to just look at the local picture near ∂S. ∎

In this way, any hierarchy $M \xrightarrow{S_1} M_1 \xrightarrow{S_2} \cdots \xrightarrow{S_n} M_n$ of M gives rise to a *sutured manifold hierarchy* $(M,\gamma) \xrightarrow{S_1} (M_1,\gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n,\gamma_n)$.

6.3. DEFINITIONS: A sutured manifold (M,γ) is *taut* if M is irreducible and R_+ and R_- are both taut surfaces in M. That is: each $R_\pm$ is incompressible, and there is no surface $(T,\partial T) \subset (M,\partial M)$, with $\chi_-(T) < \chi_-(R_\pm)$, $\partial T = \gamma$, and $[T,\gamma] = [R_\pm,\gamma]$ in $H_2(M,\eta(\gamma))$. Note this implies that $\chi_-(R_+) = \chi_-(R_-)$.

A *taut sutured manifold decomposition* is a sutured manifold decomposition $(M,\gamma) \xrightarrow{S} (M',\gamma')$ such that both (M,γ) and (M',γ') are taut sutured manifolds.

A *taut sutured manifold hierarchy* is a sutured manifold hierarchy consisting entirely of taut sutured manifolds. Note that then (M_n,γ_n) must be a collection of 3-balls with a single suture on each component.

Here are the three main theorems in sutured manifold theory:

6.4. THEOREM. *If (M,γ) is a taut sutured manifold and $\alpha \in H_2(M,\partial M)$ is non-trivial, then there is a surface $(S,\partial S) \subset (M,\partial M)$ such that $[S,\partial S] = \alpha$ and $(M,\gamma) \xrightarrow{S} (M',\gamma')$ is a taut sutured manifold decomposition.*

PROOF: The idea is to exploit 5.2 iv). Let β be the fundamental class $[R_\pm,\partial R_\pm]$ in $H_2(M,\eta(\gamma))$. Then choose k so large that $x(\alpha+(k+1)\beta) = x(\alpha+k\beta)+x(\beta)$. Then choose S to be a taut representative of the class $\alpha+k\beta$. For (M',γ') the result of this decomposition, we have $\chi_-(R'_\pm) = \chi_-(S)+\chi_-(R_\pm) = x(\alpha+k\beta)+x(\beta) = x(\alpha+(k+1)\beta) = x[R_\pm,\gamma']$, so $R'_\pm$ are norm-minimizing.

On close inspection, this argument makes little sense. For example, α is in $H_2(M,\partial M)$ not $H_2(M,\eta(\gamma))$, whereas β is in $H_2(M,\eta(\gamma))$ and indeed becomes trivial in $H_2(M,\partial M)$. So there is no way to make sense of the term $\alpha+k\beta$. This difficulty can be overcome, e.g. by generalizing the Thruston norm to "β-norms", but the proof becomes quite technical. ∎

6.5. THEOREM. *Any taut sutured manifold has a taut sutured manifold hierarchy.*

PROOF: It seems that this should be immediate from 4.9 and 6.4. The difficulty is that the surfaces S arising in 6.4 may be ∂-compressible, so in principal a partial hierarchy using such surfaces might extend indefinitely.

This problem is overcome by defining a notion of the "complexity" of a sutured manifold that is reduced during a sutured manifold decomposition and which, when trivial, terminates the hierarchy. The details are complicated. See [**Sc**, §4] or [**Ga**, §4]. ∎

6.6. THEOREM. *Suppose (M,γ) is a sutured manifold and $(M,\gamma) \xrightarrow{S} (M',\gamma')$ is a sutured manifold decomposition. If (M',γ') is taut, then so is (M,γ).*

PROOF: The proof of this is a very useful exercise for the reader. You should discover during the proof that the theorem is false as stated. There are two types of exceptions: If M is a solid torus whose boundary contains no sutures, then M is not taut, yet decomposing along a meridian disk gives a 3-ball with a single suture, which is taut. Also you must assume that no component of ∂S bounds a disk, either in S or in $\partial R_\pm$. ∎

6.7. COROLLARY. *A sutured manifold hierarchy is taut if the last term (M_n, γ_n) is taut.*

WARNING: Owing to the exceptions mentioned in the proof of 6.6, there are minor side conditions on the hierarchy that must also be satisfied.

7. Method of application

The references contain many applications of this theory. The philosophy of each proof is roughly the following: We wish to know that a particular surface S in a 3-manifold M is taut. Somehow we associate to $S \subset M$ a different situation: a surface T in a 3-manifold N which we know is taut. Using 6.5 construct a taut sutured manifold hierarchy for N. Use this hierarchy to induce a sutured manifold hierarchy on M. From the tautness of the last term (N_n, γ_n), deduce that the last term in the hierarchy of M is taut. Then apply 6.6 to deduce that S is taut.

Here are two examples.

7.1. The first example reflects well the philosophy of such proofs and is fairly elementary. The theorem is due to Gabai [Ga, 6.13], but we offer here a "defoliated" proof, which arose in a conversation with Darren Long.

Suppose $p : \widetilde{M} \to M$ is a k-fold covering map of a closed oriented irreducible 3-manifold M. There is a *transfer map* $tr : H_i(M) \to H_i(\widetilde{M})$ defined as follows: choose triangulations for $\widetilde{M}$ and M so that p is simplicial. For any α in $H_i(M)$ choose an i-cycle ζ representating it. Let $tr(\alpha)$ be the homology class represented by $p^{-1}(\zeta)$. Note that $p_*(tr(\alpha)) = k\alpha$.

THEOREM. *For x and $\tilde{x}$ the Thurston norms on $H_2(M)$ and $H_2(\widetilde{M})$ respectively, $\tilde{x}(tr(\alpha)) = kx(\alpha)$.*

PROOF: Let S be a taut representative of α in M and let $\widetilde{S} = p^{-1}(S)$. Since $\chi(\widetilde{S}) = k\chi(S)$, we have $\tilde{x}(tr(\alpha)) \leq kx(\alpha)$. To prove equality, we only need to show that $\widetilde{S}$ is taut in $\widetilde{M}$.

Let $(M, \gamma = \emptyset) \xrightarrow{S_1 = S} (M_1, \gamma_1 = \emptyset) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n)$ be a taut sutured manifold hierarchy for M. Then for $1 \leq i \leq n$, let $\widetilde{S}_i = p^{-1}(S_i)$ and $\widetilde{M}_i = p^{-1}(M_i)$. Then it's easy to see that $(\widetilde{M}, \emptyset) \xrightarrow{\widetilde{S}_1 = \widetilde{S}} (\widetilde{M}_1, \emptyset) \xrightarrow{\widetilde{S}_2} \cdots \xrightarrow{\widetilde{S}_n} (\widetilde{M}_n, p^{-1}(\gamma_n))$ is a sutured manifold hierarchy. But since M_n consists of balls with a single suture in each component, and $\widetilde{M}_n$ is a cover of M_n, $(\widetilde{M}_n, p^{-1}(\gamma_n))$ is also taut. Hence $\widetilde{S}$ is taut. ∎

The second application comes from knot theory, but requires a preliminary general theorem on "torus fillings".

7.3. Suppose M is a 3-manifold and T is a torus component of ∂M. Let σ be an essential simple closed curve in T. Then there is a homeomorphism $\varphi : \partial D^2 \times S^1 \to T$, well-defined up to isotopy, such that $\varphi(\partial D^2) = \sigma$. The manifold $M(\sigma)$ obtained by attaching $D^2 \times S^1$ to M via φ is a *filling of M at T with slope σ*.

A connected 3-manifold N with torus boundary component $T \subset \partial N$ is called a *J-cobordism* on T if $H_2(N, \partial N - T) = 0$. In other words, any surface $(S, \partial S) \subset (N, \partial N - T)$ separates. It's a consequence of Poincare duality that genus$(\partial N - T) \leq 1$. If genus$(\partial N - T) = 1$, then N has the rational homology of $T \times I$.

THEOREM (Gabai). *Let (M, γ) be a connected taut sutured manifold with $\gamma \neq \emptyset$. Suppose $T \subset \partial M$ is a torus such that $\gamma \cap T = \emptyset$ and the only J-cobordism on T contained in M is $T \times I$. Then there is at most one slope σ for which $(M(\sigma), \gamma)$ is not taut.*

PROOF: **Special case:** M is itself a J-cobordism on T.

Then M is $T \times I$, and the sutures γ lie in the torus $T' = \partial M - T$. Since (M, γ) is taut, γ consists of parallel essential curves in T'. $M(\sigma)$ is just a solid torus, so $(M(\sigma), \gamma)$ is taut unless the annuli $R_\pm = T' - \gamma$ are compressible. But this will occur only for the slope σ given by the annuli γ in T'.

General case: If M itself is not a J-cobordism on T, then construct a taut sutured manifold hierarchy of (M, γ), using always surfaces disjoint from T, until we reach a sutured manifold of the form (M_n, γ_n) with $H_2(M_n, \partial M_n - T) = 0$. Since M_n is a J-cobordism on T, $M_n = T \times I$. The torus $T' = \partial M_n - T$ arose from sutured manifold decompositions of M, so it must contain some sutures. From 6.7 $(M(\sigma), \gamma)$ is taut if $(M_n(\sigma), \gamma_n)$ is taut. Now just apply the special case to M_n. ∎

7.3. Any knot k in S^3 is the boundary of some connected orientable surface S, called a *Seifert surface* of the knot. The *genus* of the knot is the minimal genus of all Seifert surfaces. Figure 11 shows a genus one Seifert surface of the trefoil knot.

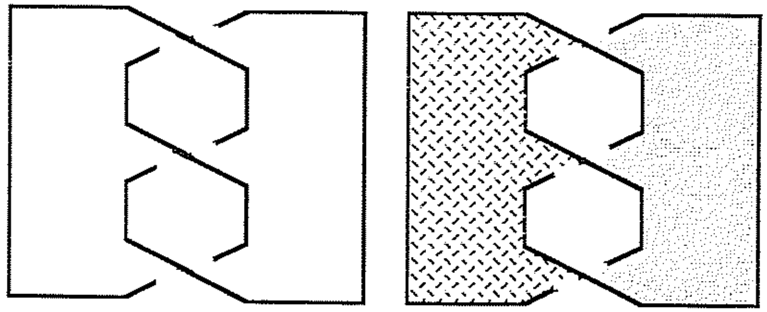

Figure 11

Suppose k_0 and k_1 are two knots in S^3, separated by a 2-sphere. Denote $k_0 \cup k_1$ by K. Suppose $b : I \times I \to S^3$ is an imbedding such that $b^{-1}(k_0) = \{0\} \times I$ and $b^{-1}(k_1) = \{1\} \times I$. Then the knot K' obtained from K by removing $b(\partial I \times I)$ and replacing it with $b(I \times \partial I)$ is called the *band sum* of k_0 and k_1, and is denoted $k_0 \#_b k_1$. See Figure 12.

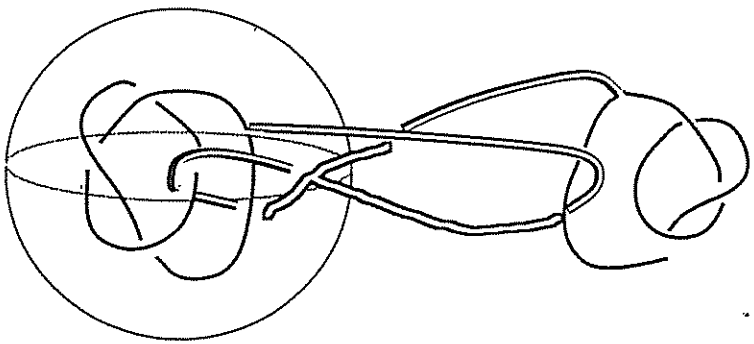

Figure 12

THEOREM (Gabai, Scharlemann). $\text{genus}(k_0 \#_b k_1) \geq \text{genus}(k_0) + \text{genus}(k_1)$.

PROOF: The proof given here is essentially Gabai's. Let L be a small circle that links the band, i.e., L bounds a disk D intersecting $b(I \times I)$ in some $\{\text{point}\} \times I$. Let $M = S^3 - (K' \cup L)$. M is irreducible and has only torus boundary components, so we may view M as a taut sutured manifold.

It's not hard to show that M satisfies the J-cobordism condition of 7.2 for J-cobordisms on $\partial\eta(L)$. In M, K' bounds an orientable surface. Let S be one of minimal genus in M. S is split by D into Seifert surfaces for k_0 and k_1, so $\text{genus}(S) \geq \text{genus}(k_0) + \text{genus}(k_1)$.

Let M_1 denote the taut sutured manifold obtained by decomposing M along S. It's not hard to see that the manifold $M_1(\sigma)$ obtained by filling in a solid torus $\partial\eta(L)$ with slope ∂D contains a reducing sphere. In particular $M_1(\sigma)$ is not taut. Now fill in $\partial\eta(L)$ with a meridian of $\eta(L)$. By 7.2 the sutured manifold $M_1(\tau)$ is taut, so S is also of minimal genus for K' in $S^3 - K' = M(\tau)$. ∎

7.4. This has an important corollary. The original proof of this corollary, which predated sutured manifold theory, required a complicated combinatorial argument.

COROLLARY. *If $k_0 \#_b k_1$ is the unknot, then k_0 and k_1 are the unknot, and the band crosses the 2-sphere separating k_0 from k_1 exactly once.*

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