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Uniqueness of bridge surfaces for 2-bridge knots

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Abstract

Any 2-bridge knot in S^3 has a bridge sphere from which any other bridge surface can be obtained by stabilization, meridional stabilization, perturbation and proper isotopy.

1. *Introduction*

One of the earliest approaches to understanding the topology of closed 3-manifolds was to divide the 3-manifold into two very simple pieces, called handlebodies, and focus on the properties that can be discerned from the way in which the two handlebodies are glued together. This naive way of decomposing the 3-manifold, called a Heegaard splitting, proved only modestly helpful, until the breakthrough work of Casson and Gordon [**CG**] established that a manifold without incompressible surfaces admitted a splitting (called a *strongly irreducible* splitting) that could for many purposes be manipulated much like an incompressible surface. This allowed some of the combinatorial theory of intersecting surfaces, which had been quite successful in describing 3-manifolds, to be extended also to those manifolds (called non-Haken manifolds) that did not contain incompressible surfaces. Some other more recent applications of Heegaard theory have been to Heegaard Floer homology (see [**OS**]) and topological quantum field theory (see [**Wi**]).

One interesting and surprisingly difficult problem is to determine to what extent Heegaard splittings for a particular manifold are unique. One of the earliest results was that of Waldhausen [**Wa**] who proved that S^3 has a unique Heegaard splitting up to stabilization. In [**BoO**], Bonahon and Otal proved that the same is true of lens splaces (manifolds with a genus one Heegaard surface). A later proof [**RS1**] made use of the fact that any two weakly incompressible Heegaard splittings of a manifold can be isotoped to intersect in a nonempty collection of curves that are essential on both Heegaard surfaces.

Much less studied has been the natural analogue to Heegaard splitting in the theory of links in 3-manifolds. (By link, we include the possibility that K has one component, i.e.

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a knot is a link.) Consider a link K in a closed orientable 3-manifold M with a Heegaard surface P (i.e. $M = A \cup_P B$ where A and B are handlebodies) and require that each arc of $K - P$ is P-parallel in the handlebody A or B in which it lies. We say that K is in *bridge position* with respect to P and that P is a *bridge surface* for the pair (M, K). Beyond the philosophical analogy between Heegaard splittings for 3-manifolds and bridge surfaces for links in 3-manifolds, notice that there is also this precise connection: If P is a bridge surface for a link K in M, then the cover $\hat{P}$ of P in the 2-fold branched cover $\hat{M}$ of M is a Heegaard surface for the manifold $\hat{M}$.

Questions about the structure of Heegaard splittings on 3-manifolds often have analogies with questions about bridge surfaces. For example, it is natural to ask whether there are pairs (M, K) that have a unique bridge surface, up to some obvious geometric operations analogous to Heegaard stabilization. In [Ot1] Otal proved that this is true for bridge spheres of the unknot (this was extended to bridge surfaces in [HS2]). In [Ot2] Otal proves the same for bridge spheres of 2-bridge knots. Here we use the philosophy of [RS1] to extend [Ot2] to all bridge surfaces of 2-bridge knots. (And presumably for 2-bridge links as well, though we do not pursue that here, because of the technical obstacle that the theory in [**STo**] so far has not been explicitly extended to 3-manifolds with non-empty boundary. Compare [**RS2**] to [**RS1**].) This result can be viewed as the analogue for bridge surfaces of the result of Bonahon and Otal mentioned above.

Our approach will be analogous to that of [**RS1**], working from the central result of [**STo**]: in the absence of incompressible Conway spheres, two c-weakly incompressible bridge surfaces can be properly isotoped to intersect in a non-empty collection of closed curves, each of which is essential (including non-meridional) in both surfaces. Here is an outline: after introducing notation and definitions (Section 2) we discuss in Section 3 some simple ways in which one bridge surface can be changed to another and how to detect the change via the topology of the bridge surface complements. Changes of this sort won't be considered particularly significant because they are so simple. In Section 4 we focus on 2-bridge knots, exploiting the fact that intersection only along essential curves guarantees that in the standard 2-bridge sphere all intersection curves are parallel. This implies that parts of any proposed alternate bridge surface are parallel to parts of the standard bridge surface. The parallelism can then be used to lower the number of curves of intersection of the two surfaces; uniqueness then follows by a careful case-by-case analysis.

Just as Bonahon and Otal's work on Heegaard splittings of Lens spaces was the first step towards the understanding of Heegaard splittings of Seifert manifolds [**MS**] (including important examples of non-uniqueness of Heegaard splittings, see [**BZ**]) it is natural to ask whether the approach here can be extended to larger classes of knots, e.g. Montesinos knots.

2. *Definitions and notation*

If X is any subset of a 3-manifold M and K is a 1-manifold properly embedded in M, let $X_K = X - K$. A disk $D \subset M$ that meets K exactly once is called a *punctured disk*. If F is an embedded surface in M transverse to K, a simple closed curve on F_K is *essential* if it doesn't bound a disk or a punctured disk on F_K. An embedded disk $D \subset M_K$ is a *compressing disk* for F_K if $D \cap F_K = \partial D$ and ∂D is an essential curve in F_K. A *cut-disk* for F_K is a punctured disk D^c in M_K such that $D^c \cap F_K = \partial D^c$ and ∂D^c is an essential curve in F_K. A possibly punctured disk D^* that is either a cut disk or a compressing disk will be called a *c-disk* for F_K. The surface F_K is called essential if it has no compressing disks (it may have cut-disks), it is not a sphere that bounds a ball in M_K and it is not ∂-parallel in $M - \eta(K)$ where $\eta(K)$ is a regular open neighbourhood of K.

A properly embedded arc $\alpha \subset F_K$ is *inessential* if there is a disk on F_K whose boundary is the endpoint union of α and a subarc of ∂F. Otherwise α is *essential*. A *∂-compressing disk* for F_K is an embedded disk $D \subset M$ with an interior disjoint from F_K such that ∂D is the endpoint union of an essential arc of F_K and an arc lying in ∂M.

Any term describing the compressibility of a surface can be extended to account not only for compressing disks but also c-disks. A surface in M that is transverse to K will be called *c-incompressible* if it has no c-disks. A surface F in M is called a *splitting surface* if M can be writen as the union of two 3-manifolds along F. If F is a splitting surface for M, we will call F_K *c-weakly incompressible* if any pair of c-disks for F_K on opposite sides of the surface intersect. If F_K is not c-weakly incompressible, it is *c-strongly compressible*.

A properly embedded collection of arcs $T = \bigcup_{i=1}^{n} \alpha_i$ in a compact 3-manifold is called *boundary parallel* if there is a collection $E = \bigcup_{i=1}^{n} E_i$ of embedded disks, so that, for each $1 \leqslant i \leqslant n$, ∂E_i is the end-point union of α_i and an arc in the boundary of the 3-manifold. A standard cut-and-paste arguments shows that if there is such a collection, there is one in which all the disks are disjoint. If the manifold is a handlebody A, the arcs are called *bridges* and disks of parallelism are called *bridge disks*. Let M be a closed irreducible 3-manifold and let P be a Heegaard surface for M decomposing the manifold into handlebodies A and B. A link K is in bridge position with respect to P if each collection of arcs $A \bigcap K$ and $B \bigcap K$ is a collection of bridges. We say that P is a bridge surface for the pair (M, K) and the triple $(M; P, K)$ is a bridge presentation of $K \subset M$.

Two disjoint surfaces $F, S \subset M$ transverse to K will be called parallel if they cobound a product region and all arcs of the link in that region can be isotoped to be vertical with respect to the product structure. F is properly isotopic to S if there is an isotopy from F to S so that F remains transverse to K throughout the isotopy, i.e. the isotopy of F_K to S_K is proper in M_K. Unless otherwise stated, all isotopies will be proper isotopies.

3. *New bridge surfaces from old*

Given a bridge surface P for (M, K), it is easy to construct more complex bridge surfaces for (M, K) from P. There are three straightforward ways to do this. The first is easiest: simply add a trivial 1-handle to one of the handlebodies, say A. This creates a dual 1-handle in B. The new bridge surface, P' is said to be *stabilized* and it is characterized by the presence of compressing disks for P', one in A and one in B, that intersect in exactly one point.

A second way to construct a more complicated bridge surface is almost as easy to see: Suppose there are a pair of bridge disks $E_A \subset A$ and $E_B \subset B$ so that the arcs $E_A \bigcap P$ and $E_B \bigcap P$ intersect precisely at one end. Then K is said to be *perturbed* with respect to P (and vice versa), and E_A, E_B are called *cancelling disks* for K. (This is one of two cases of the notion of "cancellable" bridges, as defined by Hayashi and Shimokawa in [**HS2**]. The other case occurs when a component of K is in 1-bridge position, and both bridges, and so a whole component, can be simultaneously isotoped into the bridge surface.) The word perturbed is used because one way a bridge presentation with this property can be obtained is by starting with any bridge presentation for K and perturbing K near a point of $K \bigcap P$, introducing a minimum and an adjacent maximum. The following lemma shows this is in some sense the only way in which a perturbed link can arise.

LEMMA 3·1. *Suppose K is perturbed with respect to the bridge surface P. Then there is a knot K' in bridge position with respect to P, such that $|K' \cap P| = |K \cap P| - 2$ and K is properly isotopic to the knot obtained from K' by introducing a minimum and an adjacent maximum near a point of $K' \cap P$.*

Proof. Let E_A, E_B be the cancelling bridge disks, intersecting P in arcs α and β respectively, so that $\alpha \cap \beta = E_A \cap E_B$ is a single point $p \in P$, an end point of both α and β. A standard cut-and-paste argument shows that there is a disjoint collection of bridge disks for $K \cap A$ so that the collection contains E_A. In fact:

Claim. There is a disjoint collection Δ_A of bridge disks for $K \cap A$ so that $E_A \in \Delta_A$ and $\Delta_A \cap \beta = \partial\beta$.

We begin with a disjoint collection and redefine it so as to eliminate all intersection points with the interior of β. The proof is by induction on the number of points in $\Delta_A \cap interior(\beta)$. If the intersection is empty, there is nothing to prove. Otherwise, suppose that q is the closest point of $\Delta_A \cap \beta$ to p in *interior*(β), and let β' be the subsegment of β between q and p. Suppose $E' \neq E_A$ is the bridge disk containing q. Then a regular neighborhood of $E' \cup \beta' \cup E_A$ has boundary consisting of two disks – one parallel to E' and the other a new bridge disk for the bridge $E' \cap K$ that is disjoint from all other bridge disks and intersects β in one fewer point. This provides the inductive step, establishing the claim.

Following the claim, let $E' \neq E_A$ be the bridge disk in Δ_A that is incident to the opposite end of β from p; following the claim E', like E_A, is disjoint from the interior of β. Use E_B to (non-properly) isotope the arc $K \cap E_B$ to β and push it through P. This reduces the number of points in $K \cap P$ by two, but P is still a bridge surface for the knot. It's clear that $K \cap B$ still consists of bridges, since all we've done is remove one. The change in $K \cap A$ is to attach the bridge disk E' to E_A by a band, and the result is clearly still a disk. It's easy to see that the original positioning of K is properly isotopic to a perturbation of the new positioning of K with respect to P.

Here is a third way to produce a new bridge surface for (M, K), called *meridional stabilization*. Begin with a bridge presentation $M = A \cup_P B$ of K and suppose there is a component K_0 of K that is not in 1-bridge position with respect to P. Let β be a bridge in $K_0 \cap B$ and let A' be the union of A together with a neighbourhood of β. Let $P' = \partial A'$ and let B' be the closed complement of A' in M. The decomposition $M = A' \cup_{P'} B'$ is a Heegaard splitting, indeed a stabilization of $M = A \cup_P B$ since a meridian for A' dual to β intersects the remnants of a bridge disk for β in B' in a single point. Moreover, K is in bridge position with respect to P'. It is obvious that $K \cap B'$ is a collection of bridges, since $K \cap B$ was. And the new component of $K \cap A'$ has, as a bridge disk, the union of two bridge disks of $K \cap A$ attached together by a band running along β.

LEMMA 3·2. *A bridge surface P' for K is meridionally stabilized if and only if there is a cut-disk in A' and a compressing disk in B' (or vice versa) that intersect in exactly one point.*

Proof. If P' is constructed by meridional stabilization, as described above, then, as we have seen, a meridian disk in A' dual to β is a cut disk for A' that intersects the remnants of a bridge disk for β in a single point.

Conversely, suppose there is a cut disk $E_A \subset A'$ for A' and a compressing disk $E_B \subset B'$ that intersect in a single point. Then P' is the stabilization of the Heegaard surface P obtained by cutting A' along E_A.

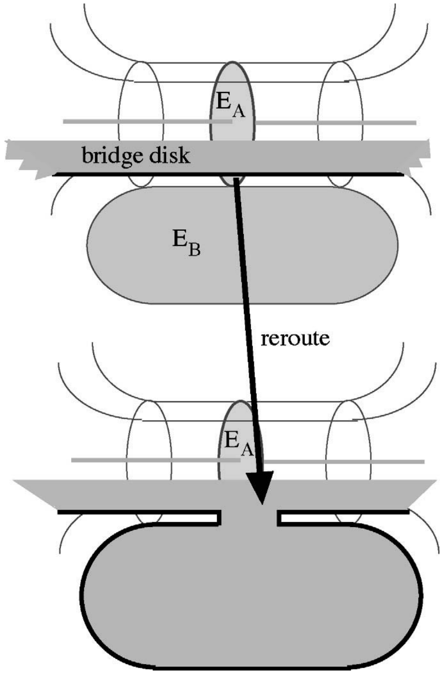

Fig. 1.

Claim. K is in bridge position with respect to P.

A standard cut and paste argument shows that the bridge disks for $K \cap B'$ can be taken to be disjoint from E_B. They can also be taken to be disjoint from ∂E_A, for any time a bridge disk for K in B' crosses ∂E_A, one can reroute it around ∂E_B, adding a copy of the disk E_B to the bridge disk, to get a bridge disk which intersects ∂E_A fewer times (see Figure 1). Once all bridge disks for $K \cap B'$ are disjoint from E_A, they persist when P' is surgered along E_A. So all components of $K \cap B$ have bridge disks, except possibly the new bridge β that is produced in B, the bit of K that runs from one copy of E_A (after the cut) to the other. But E_B itself provides a bridge disk for β.

A similar argument exhibits bridge disks in A: a standard cut and paste argument shows that there is a complete collection of bridge disks for $K \cap A'$ that intersects E_A in a single arc, running from the point $K \cap E_A$ to ∂E_A. When A' is cut apart by E_A to produce A, the bridge disk for the component of $K \cap A'$ that intersects E_A is divided by this arc into bridge disks for the two resulting components of $K \cap A$, establishing the claim.

With the claim established, it is easy to see that P' is a meridional stabilization of P along β.

Here is yet a fourth way to construct one bridge surface from another. It will be useful here to extend, in an obvious way, the definition of bridge surface to links in compact orientable 3-manifolds with boundary. Suppose M is a compact orientable 3-manifold. A connected closed surface $P \subset M$ is a bridge surface for $K \subset M$ if P is a Heegaard surface for M (that is, the complement of P consists of two compression bodies C_1, C_2 and $P = \partial_+ C_i$, $i = 1, 2$) and K intersects each complementary compression body in a collection of boundary parallel arcs.

With that clarifying extension, suppose K_- is a link (possibly empty) in a 3-manifold N that has a torus boundary component $\partial_0 N$. Let P be a bridge surface for K_- in N; that is,

P divides N into two compression bodies, and K_- intersects each of them in a collection of boundary-parallel arcs. Fill $\partial_0 N$ with a solid torus W whose core is a new curve K_0. Then P still divides $M = N \cup_{\partial_0 N} W$ into two compression bodies and K_- still intersects each compression body in a collection of boundary-parallel arcs. Moreover, the core curve K_0 is isotopic in W to a curve on $\partial W = \partial_0 N$, so K_0 is isotopic in M rel K_- to a curve on P. Perturbing K_0 slightly makes P a bridge surface for all of $K = K_- \cup K_0$ in M. If a component of a link K in bridge position with respect to P in M can be constructed in this way, then we say that the component is *removable*.

LEMMA 3·3. *Suppose P is a bridge surface for a link $K \subset M$. Then a component K_0 of K is removable if and only if K_0 can be isotoped rel $K_- = K - K_0$ so that K_0 lies on P and there is a meridian disk of one of the two compression bodies that is disjoint from K_- and intersects $K_0 \subset P$ in a single point.*

Proof. One direction is fairly straightforward: if K_0 is removable then, in the construction above, K_0 can be isotoped to a longitude of ∂W, i.e. to a curve in ∂W that intersects a meridian disk μ of W in a single point. That is, the wedge of circles $K_0 \vee \partial\mu \subset \partial W = \partial_0 N$. Let C be the compression body of $N - P$ on which $\partial_0 N = \partial W$ lies. Then, using the structure of the compression body, there is a proper embedding of $(K_0 \vee \partial\mu) \times I$ into $C - K_-$, with one end of $(K_0 \vee \partial\mu) \times I$ on ∂W and the other end on P. The end on P then describes an embedding of K_0 into P that intersects the meridian disk $\mu \cup (\partial\mu \times I)$ of the compression body $C \cup_{\partial_0 N} W$ in a single point.

The other direction uses the "vacuum cleaner trick": suppose that P is a bridge surface for a link K in M, that a component K_0 of K has been *isotoped rel* K_- to lie on P, and that μ is a meridian disk for one of the complementary compression bodies C so that μ is disjoint from K_- and μ intersects K_0 is a single point. Picture the dual 1-handle to μ in C as a vacuum-cleaner hose, and use it to sweep up all of $K_0 - \eta(\partial\mu) \subset P$. Afterwards, μ is the meridian of a solid torus that is a boundary-summand of C, a solid torus for which K_0 is a longitude. Push K_0 to the core of this solid torus and remove a thin tubular neighbourhood W of K_0 from the solid torus. This changes the solid torus to *torus* $\times I$, with the result that $C_- = C - W$ is still a compression body. Moreover, $K_- \cap C_-$ remains a collection of boundary-parallel arcs.

LEMMA 3·4. *If a bridge surface for K is stabilized then any* 1*-bridge component of K is removable.*

Somewhat conversely, suppose a component K_0 for K is removable, with P, K, K_0 and meridian disk μ as defined in the proof of Lemma 3·3 *above. Suppose further that there is a meridian disk λ for the other compression body so that λ is disjoint from K_- and $|\mu \cap \lambda| = 1$. Then P is stabilized.*

Proof. Suppose a bridge surface P for K is obtained by stabilizing the bridge surface P' for K, and suppose K_0 is a 1-bridge component of K. Let C_1, C_2 be the compression body complementary components of P'. That is, $|P \cap K_0| = |P' \cap K_0| = 2$, and P' divides K into two boundary-parallel arcs $\tau_i = C_i \cap K$, $i = 1, 2$. Let D_1, D_2 be bridge disks for τ_1, τ_2 in C_1, C_2 respectively. By general position, we can assume that the arcs $D_1 \cap P$, $D_2 \cap P$ have interiors that are disjoint near their end points (though there may be many intersections of their interiors away from the end points). Stabilize P' to P by attaching a 1-handle to C_2 via an arc α in D_1 near and parallel to $\tau_1 \subset \partial D_1$. Then D_2 together with the rectangle in D_1 lying between α and τ_1 describes an isotopy of K_0 to P'. A cocore of the 1-handle that was

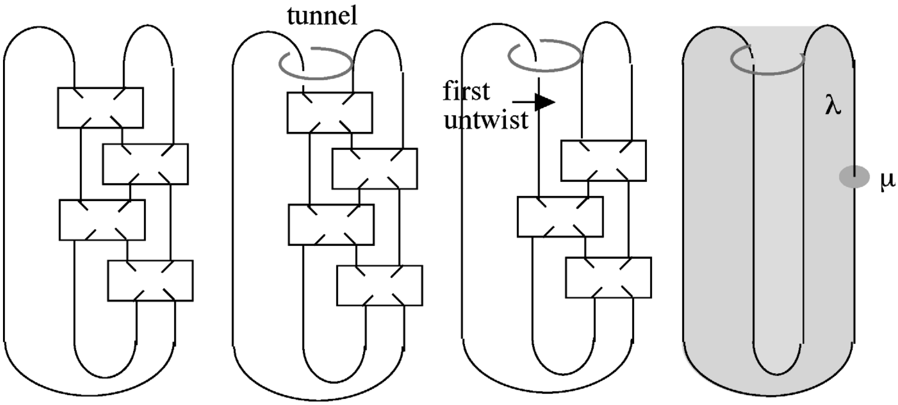

Fig. 2.

attached to C_2 is a meridian for one of the stabilized compression bodies. Via Lemma 3·3, μ exhibits K_0 as a removable component of K for the splitting surface P.

Now consider the other direction, with meridian disks $\mu \subset C_1$, $\lambda \subset C_2$, component $K_0 \subset P$ and $|K_0 \cap \mu| = 1 = |\lambda \cap \mu|$ as given in the statement of the lemma. By general position, we can assume that K_0 and λ do not intersect near μ. Move K_0 into 1-bridge position by pushing a small segment of K_0 into the interior of C_2 near μ and the interior of the rest of K_0 into the interior of C_1. Then K_0, hence all of K, is disjoint from both meridian disks λ and μ, which then exhibit that P is stabilized.

Example. Suppose K is a 2-bridge knot in S^3 and P is a Heegaard surface for the complementary 3-manifold $N = S^3 - \eta(K)$. Then either P is stabilized or it is the boundary of a regular neighbourhood of the union of the knot and a single arc, and the arc is one of six standard types (see [**Ko1**, **Ko2**, **GST**]). Each of the six types of arcs (called tunnels) has the property that, once a regular neighbourhood of the arc is added, then, up to isotopy, the regular neighbourhood no longer depends on which 2-bridge knot we started with – indeed, we could have started with the unknot. See Figure 2. In particular, there is a meridian of the complementary handlebody that intersects a meridian disk dual to the knot in a single point. Following Lemma 3·4 we then have:

COROLLARY 3·5. *Suppose P is any bridge surface for a 2-bridge knot $K \subset S^3$. If K is removable with respect to P, then P is stabilized.*

In the proof of our main theorem we will use the following known results.

LEMMA 3·6. [**STo**, lemma 3·1] *Let A be a handlebody and let $(T, \partial T) \subset (A, \partial A)$ be a collection of bridges in A. Suppose F is a properly embedded surface in A transverse to T that is not a union of unpunctured disks, once-punctured disks and twice-punctured spheres. If F_T is incompressible in A_T then $\partial F \neq \emptyset$ and F_T is ∂-compressible.*

LEMMA 3·7. [**STo**, lemma 3·6] *Suppose P and Q are disjoint bridge surfaces for a link $K \subset M$, decomposing M as $A \cup_P B$ and $X \cup_Q Y$ respectively. Suppose furthermore that $Q_K \subset A_K$ and P_K has a c-disk in A_K that is disjoint from Q_K, then either P_K is c-strongly compressible or $M = S^3$ and K is empty or the unknot.*

THEOREM 3·8. [**STo**, corollary 6·7] *Suppose P and Q are bridge surfaces for a link $K \subset M$ and P_K and Q_K are both c-weakly incompressible in M_K. If there is no incompressible*

Conway sphere for K in M then P_K can be properly isotoped so that P_K and Q_K intersect in a non-empty collection of curves that are essential on both surfaces.

THEOREM 3·9. [**To**] *Suppose, for a link $K \subset M$, M contains a c-strongly compressible bridge surface Q that is not stabilized, meridionally stabilized or perturbed. Then either:*

(i) *M contains a surface F transverse to K so that F_K is essential in M_K; or*

(ii) *K contains a component K_0 that is removable.*

4. *Unique bridge surface*

Now we will focus our attention on two-bridge links in the 3-sphere. That is, for the rest of the paper, assume $S^3 = A \cup_P B = X \cup_Q Y$, K is in bridge position with respect to both P and Q and P_K is a four times punctured sphere. In particular, henceforth A will be a ball that intersects K in two trivial arcs. The ultimate goal is to show that if K is non-trivial (i.e. neither the unknot nor the unlink of two components) and Q_K is not stabilized, meridionally stabilized or perturbed, then Q_K is also a 4-times punctured sphere properly isotopic to P_K. We will use the following technical lemma and its corollary.

LEMMA 4·1. *Suppose F_K is a connected splitting surface that is properly embedded in A, so $A_K = U_K \cup_{F_K} V_K$. Further assume ∂F consists of curves that are essential in P_K, F_K is c-incompressible in V_K, but there is a ∂-compressing disk for F_K that lies in V_K. Then F_K is parallel to a subset of P_K through V_K. In particular F_K is either an annulus or a twice punctured disk.*

Proof. Let $E \subset V_K$ be the ∂-compressing disk for F_K. Let $\sigma = E \cap P_K$ and note that σ must be an essential arc on $P_K - F_K$ as otherwise F_K would be compressible in V_K. There are two cases to consider.

First suppose that both endpoints of σ lie on the same component of ∂F; call this component f. As f is an essential curve on the 4-times punctured sphere P_K, it bounds two twice punctured disks on P_K, let P' be the twice punctured disk containing σ. A regular neighbourhood of $P' \cup E$ consists of a copy of P' and two once punctured disks, D' and D'', whose boundaries lie on F_K. As F_K is c-incompressible in V_K, D' and D'' each also bound once-punctured disks in F_K. Moreover, these disks must be parallel to the once-punctured disks on F_K, since twice-punctured spheres in a handlebody can only cut off trivial arcs from trivial arcs (cf [**STo**, lemma 3·2]). Combining these parallelisms with the boundary compression gives a parallelism betweet F_K and P'.

Suppose, on the other hand, that the two endpoints of σ lie on different components of ∂F, say f and f'. As f and f' are disjoint and essential in the 4-times punctured sphere P_K, f and f' must cobound an annulus N on P_K and $\sigma \subset N$. A regular neighbourhood of $N \cup E$ then consists of a copy of N and a disk D whose boundary lies on F_K. As F_K is incompressible in V_K, ∂D also bounds a disk in F_K, a disk that is parallel to D in A_K, since A_K is irreducible. Combining this parallelism with the boundary compression gives the desired parallelism between F_K and N.

COROLLARY 4·2. *Suppose F_K is a c-incompressible connected splitting surface, not an unpunctured disk, that is properly embedded in A, and suppose ∂F consists of curves that are essential in P_K. Then F_K is P_K-parallel.*

Proof. F_K can't be a once-punctured disk, since its boundary also bounds a twice-punctured disk in P_K. Since it's c-incompressible, it's incompressible, so by Lemma 3·6, F_K must be boundary-compressible. The result follows by Lemma 4·1

THEOREM 4·3. *Let $K \subset S^3$ be a two bridge link (not a trivial knot or link) with respect to a bridge surface $P \cong S^2 \subset S^3$. Any c-weakly incompressible bridge surface for (S^3, K) is properly isotopic to P_K.*

Proof. Suppose Q is a c-weakly incompressible bridge surface, so $S^3 = A \cup_P B = X \cup_Q Y$. P is also c-weakly incompressible. Indeed, disjoint essential curves in the 4-punctured sphere P are necessarily parallel in P_K, and so a c-strong compressing pair would provide a splitting sphere for K, contradicting the assumption that K is not a trivial link. By Theorem 3·8 we may isotope P_K so that $P_K \cap Q_K \neq \emptyset$ and all curves of $P_K \cap Q_K$ are essential on both P_K and Q_K. Furthermore assume that the number of components of intersection $|P_K \cap Q_K|$ is minimal under these restrictions. We will denote by Q_K^A and Q_K^B the surfaces $Q_K \cap A$ and $Q_K \cap B$ respectively. Similarly we will denote by P_K^X and P_K^Y the surfaces $P_K \cap X$ and $P_K \cap Y$.

Claim 1. *At least one of Q_K^A or Q_K^B has a P-parallel component*

Q_K is not a twice-punctured sphere, since K is not the unknot. Thus there are c-disks for Q_K in both X and Y. First we will reduce to the case that there are c-disks for Q_K in both X and Y that are both disjoint from P_K.

If there aren't such c-disks, then, with no loss of generality, there is a c-disk $D_Y^* \subset Y$ for Q_K so that $|P \cap D_Y^*| > 0$ is minimal among all c-disks for Q_K in Y. If the intersection contains any simple closed curves, let α be an innermost one on D_Y^* bounding a possibly punctured disk $D_\alpha^* \subset D_Y^*$. If α were inessential in P, then a c-disk with fewer intersection curves could have been found, so α is essential in P_K. Note that as P_K is a 4-times punctured sphere and all curves of $P_K \cap Q_K$ are essential in P_K, all the curves must be parallel on P_K and are all also parallel to α. Let $N \subset P_K$ be the annulus between α and an adjacent curve of $P_K \cap Q_K$. Then by slightly isotoping the possibly punctured disk $N \cup D_\alpha^*$ we obtain c-disk for Q_K that is disjoint from P_K contradicting the choice of D_Y^*. Thus we may assume that $D_Y^* \cap P_K$ consists only of arcs. An arc of $D_Y^* \cap P_K$ that is outermost on D_Y^* cuts off a disk in Y_K that ∂-compresses Q_K^A, say, to P_K. By Lemma 4·1, Q_K^A has a component that is P_K-parallel, establishing the claim in this case.

So now assume that there are c-disks $D_Y^* \subset Y$ and $D_X^* \subset X$ for Q_K and both are disjoint from P_K. If one disk lies in A and the other in B, then the disks would have disjoint boundaries, contradicting the assumption that Q_K is c-weakly incompressible. So these c-disks must lie on the same side of P_K. Suppose without loss that they both lie in A. Then, since Q_K is c-weakly incompressible, Q_K^B must be c-incompressible in B. But by Corollary 4·2, this implies that Q_K^B has a P_K-parallel component, again establishing the claim.

Following the claim, suppose with no loss of generality that Q_K^A has a P_K-parallel component. In this case Q_K^A must be connected, for otherwise a component of Q_K^A could be isotoped across P_K reducing $|P_K \cap Q_K|$. As all components of $P_K - Q_K$ are annuli or twice punctured disks, Q_K^A is also either an annulus or a twice punctured disk. Without loss of generality, assume Q_K^A is parallel to P_K^X (through the region $A \cap X$). See Figure 3.

Suppose Q_K^B were c-compressible into Y_K with a c-disk D^*. Isotope P_K^X across Q_K^A so that $P_K \subset Y_K - D^*$. By Lemma 3·7 this would imply that Q_K is c-strongly compressible, a contradiction to our hypothesis. Thus we conclude that Q_K^B is either c-incompressible or c-compresses only into X_K. A similar argument with the roles of P and Q switched shows that P_K^Y does not c-compress into B_K.

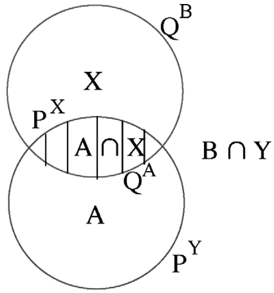

Fig. 3.

Case 1. *Q_K^A consists of a single P-parallel twice punctured disk*

This in particular implies that both P_K^X and P_K^Y consist of single twice-punctured disks.

Suppose first that P_K^Y is c-compressible in Y with c-disk D^*. As already shown $D^* \subset A$. Without loss of generality we may assume $\partial D^* = (P_K \cap Q_K)$ so D^* is also a c-disk for Q_K^A lying in Y_K. As Q_K is c-weakly incompressible, Q_K^B is either c-incompressible or also c-compresses in Y_K. As we have already eliminated the later option, Q_K^B must be c-incompressible and so, by Corollary 4·2, Q_K^B is P_K-parallel. Thus Q_K^B is a twice punctured disk so Q_K is also a 4-times punctured sphere. In summary, if P_K^Y is c-compressible, then Q_K is also a 4-times punctured sphere and Q_K^B is c-incompressible. So, by possibly switching the names of P and Q, we may henceforth assume that P_K^Y is c-incompressible.

As P_K^Y is c-incompressible, by Lemma 3·6 it must be ∂-compressible. Let E be the boundary compressing disk and note that $E \cap Q_K$ is an arc essential on $Q_K - P_K$ as otherwise P_K^Y would be compressible. Thus, by changing our point of view, we can conclude that Q_K^A or Q_K^B is ∂-compressible in A or B respectively to P_K^Y.

Suppose Q_K^B is ∂-compressible to P_K^Y. As Q_K^B is c-incompressible in Y, Lemma 4·1 implies that Q_K^B is parallel to P_K^Y. Combining this with parallelism between Q_K^A and P_K^X gives the desired isotopy between P_K and Q_K.

Suppose Q_K^A is ∂-compressible into P_K^Y. Since P_K^Y is a c-incompressible splitting surface for Y, it follows from Lemma 4·1 that P_K^Y is parallel to Q_K^A, i.e. Q_K^A is isotopic to both P_K^X and P_K^Y. In particular P_K can be properly isotoped to lie in either X_K or Y_K. By Lemma 3·7 this implies that Q_K^B must be c-incompressible in B_K, for if Q_K^B has a c-disk lying in X_K (say) we could isotope P_K to lie in X_K and be disjoint from this c-disk. By Corollary 4·2 this implies that Q_K^B is parallel to one of P_K^X or P_K^Y. As Q_K^A is parallel to both P_K^X and P_K^Y we conclude that P_K and Q_K are properly isotopic.

Case 2. *Q_K^A is a single P_K^X-parallel annulus*

We will show, by contradicition, that this case does not arise. In this situation P_K^X is a single annulus and P_K^Y consists of two twice-punctured disks. See Figure 4. Recall that we have already shown that Q_K^B is c-incompressible in Y_K and P_K^Y is c-incompressible in B_K.

Suppose (towards a contradiction) that P_K^Y is c-incompressible in Y. By Lemma 3·6 it must be boundary compressible. As in the previous case the ∂-compressing disk is incident to $Q_K - P_K$ in an essential arc, i.e. one of Q_K^A or Q_K^B is ∂-compressible to P_K^Y. The annulus Q_K^A can't ∂-compresses to P_K^Y, since its boundary components are on different components of P_K^Y. On the other hand, if a component of Q_K^B ∂-compresses to P_K^Y, by Corollary 4·2 it follows that Q_K^B has a twice punctured disk component parallel to one of the two components

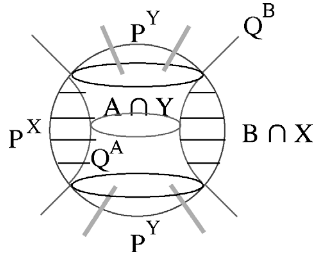

Fig. 4.

of P_K^Y. In this case $|P_K \cap Q_K|$ can be decreased by 1, and this contradicts the minimality assumption. We conclude that P_K^Y must be c-compressible in the complement of Q_K.

Suppose D^* is a c-disk for P_K^Y in the complement of Q_K. Necessarily $D^* \subset A_K$ and we may as well take ∂D^* to be one of the circles $P \cap Q$. Then D^* is also a c-disk for Q_K^A lying in $A \cap Y$. By c-weak incompressibility of Q_K any c-disks for Q_K^B would have to lie in Y_K. But we established in the beginning of this case that this is not possible so Q_K^B is in fact c-incompressible. By Lemma 3·6, Q_K^B must be boundary compressible. As we already saw, if the boundary compression is to P_K^Y, the intersection $P_K \cap Q_K$ can be reduced, so Q_K^B must be boundary compressible to the annulus P_K^X. It follows then from Lemma 4·1 that Q_K^B, like Q_K^A, is an annulus parallel to P_K^X. Then Q_K is a torus that is disjoint from K and so it cannot be a bridge surface, a contradiction.

COROLLARY 4·4. *Suppose K is a knot in S^3, 2-bridge with respect to the bridge surface $P \cong S^2$, and K is not the unknot. Suppose Q is any other bridge surface for K. Then either:*

(i) *Q is stabilized;*
(ii) *Q is meridionally stabilized;*
(iii) *Q is perturbed; or*
(iv) *Q is properly isotopic to P.*

Proof. If Q is c-weakly incompressible then Theorem 4·3 shows that Q is properly isotopic to P. If Q is c-strongly compressible, Theorem 3·9 says that either Q is stabilized, meridionally stabilized or perturbed, or K is removable with respect to the bridge surface Q, or there is a surface F transverse to K so that F_K is essential in S_K^3. The last possibility does not occur for 2-bridge knots (see [**HT**]). Corollary 3·5 shows that if K is removable with respect to Q, then Q is stabilized.

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