MAT 116 : Final Exam Study Guide

Michael Williams

Last Updated: July 27, 2010

Remarks

The final exam will consist mostly of computations, and it might also contain some short proofs or explanations. Your exam solutions are expected to be written clearly. You may apply theorems (without proof) if they were proved in class or in Brualdi's book; you may (and are encouraged to) apply combinatorial principles and formulas such as Pascal's formula, the inclusion-exlusion principle, etc. In particular, you may not use results proved on homework or in the in-class problems: it is possible that an exam question may actually be (part of) a homework problem. If you are not sure about whether or not you can apply a particular result while taking the test, you may ask me.

Practice Problems: **Homework Problems** and In-class problems.

Four basic counting principles

Principles: Addition, multiplication, subtraction, and division principles.

In-class Problems: **Problem Sheet 1**

Permutations of sets

Definitions: Permutation of a set, r–element permutation of an n–element set (i.e. r–permutation), ordinary permutation as linear permutation, circular permutation of a finite set, r–element circular permutation of an n–element set (i.e. circular r–permutation).

Formulas:

- The number of r–permutations of an n–element set is

$$P(n,r) = \frac{n!}{(n-r)!} \ .$$

- The number of circular r–permutations of an n–element set is

$$Circ(n,r) = \frac{P(n,r)}{r} = \frac{n!}{r \cdot (n-r)!} \ .$$

In-class Problems: **Problem Sheet 2**

Combinations (subsets) of sets

Definitions: Combination (subset) of a set, r–element combination (subset) of an n–element set (i.e. r–combination).

Formulas:

- The number of r–combinations of an n–element set is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!} \ .$$

  We also adopt the convention that $C(n,r) = \binom{n}{r} = 0$ whenever $r > n$.

- Pascal's formula is

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \ .$$

- The binomial formula is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \ .$$

In-class Problems: **Problem Sheet 2**

Permutations of multisets

Definitions: Multiset, types, repetition numbers, r–permutation of a multiset.

Formulas: Let S be a multiset with k types of elements. We usually write this as

$$S = \{n_1 \cdot a_1, \ldots, n_k \cdot a_k\}$$

where the a_i are the *types* and the n_i are their respective (possibly infinite) *repetition numbers*.

- The number of r–permutations of S with all *infinite* (or *sufficiently large*) repetition numbers is k^r.
- The number of permutations of S with all *finite* repetition numbers $n_1, \ldots, n_k$ such that $|S| = n = n_1 + \cdots + n_k$ is
$$\frac{n!}{n_1! \cdots n_k!} \ .$$

In-class Problems: **Problem Sheet 3**

Combinations (subsets) of multisets

Definitions: Multiset, types, submultiset (or combination), repetition numbers, r–combination of a multiset.

Formulas/Techniques: Let S be a multiset with k types of elements. We usually write this as

$$S = \{n_1 \cdot a_1, \ldots, n_k \cdot a_k\}$$

where the a_i are the *types* and the n_i are their respective (possibly infinite) *repetition numbers*.

- The number of r–combinations of S with all *infinite* (or *sufficiently large*) repetition numbers is
$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1} \ .$$
Alternatively, let $k \in \mathbb{N}$ and $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and define
$$\Sigma_{k,r} = \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_+^k : x_1 + x_2 + \cdots + x_k = r\} \ ;$$
then
$$|\Sigma_{k,r}| = \binom{r+k-1}{r} = \binom{r+k-1}{k-1} \ .$$

- When counting the number of integral solutions of

$$x_1 + \cdots + x_k = r \quad (r \in \mathbb{Z})$$

with constraints

$$x_1 \geq l_1, \ldots x_k \geq l_k \ ,$$

change variables: $y_i = x_i - l_i$, then use the standard formula.

In-class Problems: **Problem Sheet 3**

Pidgeonhole principle

Pidgeonhole Principle (Simple Form). *If $n+1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.*

In-class Problems: **Problem Sheet 4**

Pidgeonhole Principle (Strong Form). *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are distributed into n boxes, then the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ..., or the n^{th} box contains at least q_n objects.

Averaging Principle. *If the average of n nonnegative integers $m_1, m_2, \ldots, m_n$ is at least equal to r, then at least one of the integers $m_1, m_2, \ldots, m_n$ satisfies $m_i \geq r$.*

In-class Problems: **Problem Sheet 5**

The inclusion-exclusion principle

The Inclusion-Exclusion Principle. *Let S be a finite set. Suppose that $A_1, A_2, \ldots, A_m$ are subsets of S. Then*

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| \ = \ |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|$$

where the first sum is over all 1–subsets $\{i\}$ of $\{1, 2, \ldots, m\}$, the second sum is over all 2–subsets of $\{1, 2, \ldots, m\}$, the third sum is over all 3–subsets $\{i, j, k\}$ of $\{1, 2, \ldots, m\}$, and so on until the mth sum over all m–subsets of $\{1, 2, \ldots, m\}$ of which the only one is itself.

The Inclusion-Exclusion Principle (Alternative Version). *Let S be a finite set. Suppose that $\mathscr{C} = \{A_1, A_2, \ldots, A_m\}$ is a collection of subsets of S. Given a number $k \in \{1, 2, \ldots, m\}$, let $\mathscr{C}_k = \{k$–combinations of $\mathscr{C}\}$. Then*

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| \;=\; |S| + \left(\sum_{k=1}^{m} (-1)^k \sum_{\mathscr{C}_k} |A_{i_1} \cap \cdots \cap A_{i_k}| \right) .$$

Major Application:

Let $k \in \mathbb{N}$ and $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Define $\Sigma_{k,r} = \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_+^k : x_1 + x_2 + \cdots x_k = r\}$ as usual.

When counting nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_k = r \quad (r \in \mathbb{Z}_+)$$

subject to

$$0 \le x_1 \le l_1, 0 \le x_2 \le l_2, \ldots, 0 \le x_k \le l_k \ ,$$

apply the inclusion-exclusion priniciple with

- $S = \Sigma_{k,r}$
- $A_i = \{(x_1, x_2, \ldots, x_k) \in S : x_i \ge l_i + 1\}$ whenever $l_i < r$. Here, the point is that you need only define A_i whenever the inequality $x_i \le l_i$ actually *constrains* your solutions.

In-class Problems: **Problem Sheet 6**

Derangements

Definition: Derangement of an n–element set.

Formulas: If S is an n–element set ($n \in \mathbb{N}$), let D_n denote the number of derangements of S.

- The formula for D_n is

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right) .$$

- Two recurrence relations for D_n are

  (i) $D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \text{for } n \geq 3$

  (ii) $D_n = n \cdot D_{n-1} + (-1)^n \quad \text{for } n \geq 2.$

Fibonacci sequence

Definition: Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ as the recursive sequence $f_0, f_1, f_2, f_3, \ldots$ where

$$f_0 = 0 \ , f_1 = 1 \ , \ f_n = f_{n-2} + f_{n-1} \text{ for } n \geq 2 \ .$$

Closed Formula: Be able to solve the *Fibonacci recurrence relation with initial conditions* above in order to obtain the closed formula

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \ .$$

This calculation is an example of solving a Linear Homogeneous Recurrence Relation (LHRR) with constant coefficients.

Useful facts about power series (brief review)

References: Calculus textbook, **Wikipedia page on power series**.

We focus on geometric series:

1. A geometric series is an infinite series of the form

$$\sum_{n=0}^{\infty} a \cdot x^n$$

   where a is a constant.

2. The m^{th} partial sum is

$$\sum_{n=0}^{m} a \cdot x^n = a(1 + x + x^2 + \cdots + x^m) = a \left(\frac{1 - x^{m+1}}{1-x} \right) \ .$$

3. Formula:
$$\sum_{n=0}^{\infty} a \cdot x^n = \frac{a}{1-x} \quad \text{whenever } |x| < 1 \ .$$
In this class, we view series just as formal (symbolic) objects, so the convergence issues are not important.

4. Term-by-term differentiation: For example,
$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} \frac{d}{dx}[x^n] = \sum_{n=1}^{\infty} nx^{n-1} \ .$$

Generating functions

Given a sequence $h_0, h_1, h_2, \ldots$, its *generating series* (also called *generating function* in the textbook) is $\sum h_n x^n$. We reserve the term *generating function* for a closed formula for the series $\sum h_n x^n$.

In-class Problems: **Problem Sheet 7** and the first problem on **Problem Sheet 8**

Linear Homogeneous Recurrence Relations (LHRR's)

Definitions: Linear Homogeneous Recurrence Relation (LHRR), initial conditions, characteristic polynomial and characteristic equation, general solution, particular solution.

Method for solving ***constant coefficient*** *LHRR's:*

1. Put the relation into the *standard form*
$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0 \ , a_k \neq 0 \ .$$

2. Solve the *characteristic equation*
$$p(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0 \ .$$
Keep in mind that the characteristic equation arises when you substitute $h_n = x^n$ in the recurrence relation.

3. Based on the *roots of* $p(x)$, form the *general solution*.

4. Use *initial conditions* (if given) to find a *particular solution*. This will involve solving a k by k linear system, that is, a linear system of k equations and k variables.

In-class Problems: **Problem Sheet 8**

Formulas on the final exam

The following formulas will be given to you on the final exam; the formulas will be given *exactly* as they appear below. You are expected to be familiar with the *notation, meaning, and context* (as seen in this study guide) of the quantities in these formulas. It is possible that you might not need some of these formulas on the exam.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$|\{\text{permutations of } S\}| = \frac{n!}{n_1! \cdots n_k!}\ , \text{ where } S = \{n_1 \cdot a_1, \ldots, n_k \cdot a_k\} \text{ and } n = n_1 + \cdots + n_k < \infty$$

$$|\{r\text{–combinations of } S\}| = \binom{r+k-1}{r} = \binom{r+k-1}{k-1}\ , \text{ where } S = \{\infty \cdot a_1, \ldots, \infty \cdot a_k\}$$

$$\Sigma_{k,r} = \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_+^k : x_1 + x_2 + \cdots + x_k = r\} \implies |\Sigma_{k,r}| = \binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| = |S| + \left(\sum_{k=1}^{m} (-1)^k \sum_{\mathscr{C}_k} |A_{i_1} \cap \cdots \cap A_{i_k}|\right)$$

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

$$D_n = (n-1)(D_{n-1} + D_{n-2}) = n \cdot D_{n-1} + (-1)^n$$

$$\sum_{n=0}^{m} x^n = 1 + x + x^2 + \cdots + x^m = \frac{1 - x^{m+1}}{1 - x}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$