MAT 116 : Midterm Exam Solutions

Last Updated: July 14, 2010

For each problem, the first solution presented is the instructor's most anticipated solution for students to give. Any subsequent solutions are elaborations (and sometimes corrections) of various commonly occuring students' solutions.

1. How many ways are there to form a 3–letter word (i.e. sequence) containing the letter C using the letters A, B, C, D, E with repetitions of letters allowed?

   Solution 1: Use the subtraction principle.

   Let U be the set of all 3–letter words in the letters A, B, C, D, E. Since we have 5 choices for each of the 3 letters, the multiplication principle gives $|U| = 5^3$.

   Define S to be the set of words in U which contain no C's. Since we have 4 choices for each of the 3 letters, the multiplication principle gives $|S| = 4^3$.

   This problem asks us to solve for the size of the complement of S in U. By the substraction principle, this number is

   $$|U| - |S| = 5^3 - 4^3 = 125 - 64 = 61 \ .$$

   Solution 2: Use the addition principle.

   Let U be the set of all 3–letter words in the letters A, B, C, D, E. Define T to be the set of words in U which contain at least one C. Ultimately, we would like to calculate $|T|$, but it is helpful to consider certain subsets.

   Let T_1 be the set of all words in T that contain exactly 3 C's, let T_2 be the set of all words in T that contain exactly 2 C's, and let T_3 be the set of all words in T that contain exactly 1 C. It is clear that $T_i \cap T_j = \varnothing$ for any distinct $i, j \in \{1, 2, 3\}$, and $T = T_1 \cup T_2 \cup T_3$. Therefore $|T| = |T_1| + |T_2| + |T_3|$, by the addition principle.

   There is only one word in T with exactly 3 C's, namely CCC; so $|T_1| = 1$.

   Any word in T_2 is formed by choosing a position (out of a possible 3 choices) for the non-C letter, then choosing a non-C letter (out of a possible 4 choices) for that position. Therefore $|T_2| = 3 \cdot 4 = 12$, by the multiplication principle.

   Any word in T_3 is formed by choosing a position (out of a possible 3 choices) for the C, then choosing a 2–letter word (without any C's) with repetitions allowed (out of a possible $4^2 = 16$ choices). Therefore $|T_3| = 3 \cdot 4^2 = 48$, by the multiplication principle.

   Therefore, $|T| = |T_1| + |T_2| + |T_3| = 1 + 12 + 48 = 61$.

2. Seven people, including two who do not want to sit next to each other, are to be seated at a round table. How many (circular) arrangements are there?

Solution 1: Use the multiplication principle.

Suppose that we list the seven people as $P_1, P_2, \ldots, P_7$, where P_1 and P_2 do not want to sit next to each other.

A circular arrangement of the seven people in which P_1 and P_2 do not sit next to each other can be formed by

Step 1: Seat P_1 arbitrarily (as the "head of the table").

Step 2: Seat P_2 somewhere not next to P_1.

Step 3: Seat the rest of the people $P_3, P_4, \ldots, P_7$.

Since P_1 is seated first in this circular arrangement, there is effectively 1 choice for step 1.

There are $7 - 3 = 4$ choices for step 2 because there are 3 forbidden seats out of the 7 seats.

There are five remaining seats for the five remaining people $P_3, P_4, \ldots, P_7$. So step 3 amounts to a linear permutation of five people. Therefore, there are 5! choices for step 3.

By the multiplication principle, we have a total of $4(5!)$ circular arrangements.

Solution 2: Use the subtraction principle.

Suppose that we list the seven people as $P_1, P_2, \ldots, P_7$, where P_1 and P_2 do not want to sit next to each other.

Let U be the set all circular arrangements of the seven people. Let $A \subset U$ be the arrangements in which P_1 and P_2 always sit next to each other. By the subtraction principle, our final answer will be $|U| - |A|$.

By our standard formula for circular arrangements, we have $|U| = Circ(7,7) = \frac{7!}{7} = 6!$.

To compute $|A|$, we temporarily consider $\{P_1, P_2\}$ to be a single person X, count the circular arrangements of the six objects $X, P_3, P_4, \ldots, P_7$ (there are $Circ(6,6) = 5!$ such arrangements), then multiply this count by 2 (in order to account for the permutations P_1P_2 and P_2P_1). Therefore, $|A| = 5! \cdot 2$.

Therefore, the number of circular arrangements in which P_1 and P_2 do not sit next to each other is

$$|U| - |A| = 6! - (5! \cdot 2) = (6-2)(5!) = 4(5!) \ .$$

3. A bakery sells exactly 5 kinds of cookies including chocolate-chip cookies and shortbread cookies. Assuming that the bakery has at least 10 cookies of each kind, how many different options are there for a box of 10 cookies if we require that the box contains at least 1 chocolate-chip cookie and at least 2 shortbread cookies?

Solution: There are five kinds of cookies to consider. Consider the chocolate-chip cookies as the first kind, the shortbread cookies as the second kind, and let the other kinds be the third, forth, and fifth kind.

In a box of 10 cookies with the constraints of this problem, we let

x_1 denote the number of chocolate-chip cookies,

x_2 denote the number of shortbread cookies,

x_3 denote the number of the third kind,

x_4 denote the number of the forth kind, and

x_5 denote the number of the fifth kind.

The answer to our problem will be the number of integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10$$

subject to the conditions

$$x_1 \geq 1, x_2 \geq 2, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \ .$$

Change variables: $y_1 = x_1 - 1, y_2 = x_2 - 2, y_3 = x_3, y_4 = x_4, y_5 = x_5$. Therefore, the answer to our problem will be the number of nonnegative integral solutions of

$$y_1 + y_2 + y_3 + y_4 + y_5 = 10 - 1 - 2 = 7 \ .$$

Now we use the standard formula (with $k = 5$ and $r = 7$) to obtain

$$\binom{r + (k-1)}{(k-1)} = \binom{7 + (5-1)}{(5-1)} = \binom{11}{4}$$

as the number of different options for a box of cookies.

4. Show that if 51 integers are chosen from the set $S = \{1, 2, \ldots, 100\}$, then there is always a pair of chosen integers which differ by 1. **Hint:** Consider the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{99, 100\}$.

Solution 1: Use the hint and distribute the 51–element subset into the set of 50 pairs. Then apply the Pidgeonhole Principle (Simple Form). This approach is just like the approach in Application 2.

We consider the set of 50 pairs

$$Y = \big\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{99, 100\}\big\} .$$

Each pair consists of integers which differ by 1.

Suppose that X is a 51–element subset of S.

By the Pidgeonhole Principle (Simple Form), if 51 objects are distributed into 50 boxes, then some box contains at least two of the objects. In other words, no function (i.e. assignment) $X \to Y$ can be injective.

In particular, if we distribute the 51 elements (objects) of X into the 50 pairs (boxes) where $n \in X$ is assigned to the pair which contains n (i.e. $\{n, n+1\}$ (if n is odd) or $\{n-1, n\}$ (if n is even)), then at least two elements of X are assigned to the same pair in Y.

Therefore, there is always a pair of integers in X which differ by 1.

Solution 2: Do not use the hint and proceed as in the "chessmaster problem" or Application 4 of the Pidgeonhole Principle (Simple Form).

Suppose that X is a 51–element subset of S. Label the elements of X in increasing order of value. For instance, we can write $X = \{a_1, a_2, \ldots, a_{51}\}$ so that

$$1 \leq a_1 < a_2 < \cdots < a_{51} \leq 100 .$$

We want to show that $a_i = a_j + 1$ for some pair $a_i, a_j \in X$.

By adding 1 accross the above inequality, we get

$$2 \leq a_1 + 1 < a_2 + 1 < \cdots < a_{51} + 1 \leq 101 .$$

(Subtracting 1 to get $\quad 0 \leq a_1 - 1 < a_2 - 1 < \cdots < a_{51} - 1 \leq 99 \quad$ works just as well, then the rest of the solution has to be appropriately adjusted.)

Consider the collection $\mathscr{C} = \{a_1, a_2, \ldots, a_{51}, a_1 + 1, a_2 + 1, \ldots, a_{51} + 1\}$ of 102 symbols. Their values lie in the set $V = \{1, 2, \ldots, 101\}$. Now consider the function (assignment) $\mathscr{C} \to V$ defined by evaluation; equivalently, distribute the 102 objects of $\mathscr{C}$ into the 101 boxes of V in the most natural way.

By the Pidgeonhole Principle (Simple Form), at least two objects in $\mathscr{C}$ have the same value in V. Since the a_i are all distinct and the $a_j + 1$ are all distinct, we must have $a_i = a_j + 1$ for some pair $a_i, a_j \in X$. This concludes the proof.

5. How many 8–permutations are there of the 10 letters of the word CALIFORNIA?

Solution: The letters of CALIFORNIA can be viewed as the multiset

$$S = \{C, 2 \cdot A, L, 2 \cdot I, F, O, R, N\} \ .$$

To count 8–permutations of S, we consider 4 (general) types of 8–combinations, then we count the permutations of each type and apply the addition principle.

Viewing an 8–combination of S as being obtained by dropping (i.e. removing) two elements of S, we have the following (general) types:

(i) Drop $\{2 \cdot A\}$ or drop $\{2 \cdot I\}$.
Examples: $\{C, L, 2 \cdot I, F, O, R, N\}$ and $\{C, 2 \cdot A, L, F, O, R, N\}$.

(ii) Drop $\{A, I\}$.
Example: $\{C, A, L, I, F, O, R, N\}$.

(iii) Drop $\{A, \alpha\}$ or drop $\{I, \alpha\}$, where $\alpha \in \{C, L, F, O, R, N\}$.
Examples (with $\alpha = C$): $\{A, L, 2 \cdot I, F, O, R, N\}$ and $\{2 \cdot A, L, I, F, O, R, N\}$.

(iv) Drop $\{\alpha, \beta\}$, where $\{\alpha, \beta\} \subset \{C, L, F, O, R, N\}$.
Example (with $\{\alpha, \beta\} = \{C, L\}$): $\{2 \cdot A, 2 \cdot I, F, O, R, N\}$.

It is easy to see that there are only two 8–combinations of type (i). Each 8–combination here has $\frac{8!}{2!}$ permutations. Therefore, there is a total of $2 \cdot \frac{8!}{2!} = 8!$ permutations of this type.

It is easy to see that there is only one 8–combination of type (ii), and there are 8! permutations of it.

For type (iii), we obtain six 8–combinations if we drop $\{A, \alpha\}$ for the six choices for α; similarly, there are six 8–combinations if we drop $\{I, \alpha\}$ for the six choices for α. Therefore, there are only twelve 8–combinations of type (iii). Each 8–combination here has $\frac{8!}{2!}$ permutations, as in type (i). Therefore, there is a total of $12 \cdot \frac{8!}{2!} = 6(8!)$ permutations of this type.

In order to obtain an 8–combination of type (iv), we have to choose 2 elements from the 6 elements of $\{C, L, F, O, R, N\}$ to drop from S. It is now clear that the number of 8–combinations of type (iv) is $\binom{6}{2}$. Each 8–combination here has $\frac{8!}{2! \cdot 2!}$ permutations. Therefore, there is a total of $\binom{6}{2} \cdot \left(\frac{8!}{2! \cdot 2!}\right)$ permutations of this type.

By the addition principle, the number of 8–permutations of S is

$$8! + 8! + 6(8!) + \binom{6}{2} \cdot \left(\frac{8!}{2! \cdot 2!}\right) = \frac{47}{4}(8!) \ .$$