MAT 116 In-class Problems (#6)

July 12, 2010 and July 13, 2010

The Inclusion-Exclusion Principle. *Let S be a finite set. Suppose that $A_1, A_2, \ldots, A_m$ are subsets of S. Then*

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| \;=\; |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|$$

where the first sum is over all 1–subsets $\{i\}$ of $\{1, 2, \ldots, m\}$, the second sum is over all 2–subsets of $\{1, 2, \ldots, m\}$, the third sum is over all 3–subsets $\{i, j, k\}$ of $\{1, 2, \ldots, m\}$, and so on until the mth sum over all m–subsets of $\{1, 2, \ldots, m\}$ of which the only one is itself.

The Inclusion-Exclusion Principle (Alternative Version). *Let S be a finite set. Suppose that $\mathscr{C} = \{A_1, A_2, \ldots, A_m\}$ is a collection of subsets of S. Given a number $k \in \{1, 2, \ldots, m\}$, let $\mathscr{C}_k = \{k$–combinations of $\mathscr{C}\}$. Then*

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m| \;=\; |S| + \left(\sum_{k=1}^{m} (-1)^k \sum_{\mathscr{C}_k} |A_{i_1} \cap \cdots \cap A_{i_k}| \right) .$$

Problem 1. Count the permutations $i_1 i_2 \ldots i_{10}$ of $\{1, 2, \ldots, 10\}$ in which

(a) $i_1 \neq 1$.

(b) $i_1 \neq 1$ and $i_2 \neq 2$.

(c) $i_1 \neq 1$, $i_2 \neq 2$, and $i_3 \neq 3$.

Problem 2. Find the number of integers in $S = \{1, 2, \ldots, 1000\}$ that are not divisible by 5 and 6.

Problem 3. Find the number of integers in $S = \{1, 2, \ldots, 1000\}$ that are not divisible by 5, 6, and 8.

Problem 4. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

Problem 5. Count the number of 5–combinations of $\{3 \cdot a, 4 \cdot b\}$.

Problem 6. Count the number of 10–combinations of $\{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

Problem 7. What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 = 12$$

that satisfy

$$-1 \leq x_1 \leq 2 \;,\; 3 \leq x_2 \leq 7 \;,\; \text{and} \;\; 0 \leq x_3 \leq 5 \;?$$