MAT 145 : Homework Hints

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Homework 3 Hints

Problems from Crossley's Book

4.1 All that you need is the definition of connected (and disconnected) space. A proof by contradiction would be efficient.

4.8 To warm up, prove the problem for the $n = 3$ case. Then figure out how to give a proof by induction for the general case.

By definition of Hausdorff space, there are neighborhoods U_1, U_2 of x_1, x_2, respectively, so that $U_1 \cap U_2 = \varnothing$; so the problem is solved for the $n = 2$ case.

To extend this to the $n = 3$ case, use the definition of Hausdorff space to get disjoint neighborhoods of x_1 and x_3; do this again for the pair x_2 and x_3. Then form various intersections to obtain a collection of open sets V_1, V_2, V_3, each containing one, and exactly one, of the points x_1, x_2, x_3.

4.10 A little formality will be useful here. Let $L = \mathbb{R} \cup \{0'\}$, and consider the function $p : L \to \mathbb{R}$ defined by the formulas $p(0') = 0$, and $p(t) = t$ if $t \in \mathbb{R}$.

Then $U \subset L$ is open if $p(U) \subset \mathbb{R}$ is open (in the usually topology). The problem requires you to show that the collection

$$\mathcal{T}_L = \{U \subset L : p(U) \subset \mathbb{R} \text{ is open}\}$$

is a topology on L. This means that you need to show that $\varnothing \in \mathcal{T}_L$, $L \in \mathcal{T}_L$, arbitrary unions of elements of $\mathcal{T}_L$ are in $\mathcal{T}_L$, and finite intersections of elements of $\mathcal{T}_L$ are in $\mathcal{T}_L$.

You may use the following facts from basic set theory (without proof): Let $f : X \to Y$ be a function. Then

- $f(\bigcup O_\alpha) = \bigcup f(O_\alpha)$ for any collection $\{O_\alpha\}$ of subsets of X.
- $f(A \cap B) \subset f(A) \cap f(B)$ for any pair of subsets A and B of X.

The first fact about unions will help you to show that arbitrary unions of elements in $\mathcal{T}_L$ are in $\mathcal{T}_L$.

The second fact will be useful for finite intersections. Keep in mind that the function $p : L \to \mathbb{R}$ is very special: it is the identity function except at 0 and $0'$. You are recommended to prove the following

Lemma. *For any two subsets $A, B \subset L$, we have*

$$p(A \cap B) = p(A) \cap p(B) \quad \text{or} \quad p(A \cap B) = (p(A) \cap p(B)) - \{0\} .$$

Then figure out a way to apply this lemma to show that if $A, B \in \mathcal{T}_L$, then $A \cap B \in \mathcal{T}_L$.

Problems from Hatcher's Notes

15(b) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at $x \in \mathbb{R}$ on the right** if and only if $\lim_{\epsilon \to 0^+} f(x + \epsilon) = f(x)$.

This means that for every $\epsilon' > 0$, there exists a $D > 0$, so that $|f(x+\epsilon) - f(x)| < \epsilon'$ for every $0 < \epsilon < D$. To relate this to the "lower limit topology" (a.k.a "half-open" topology) on $\mathbb{R}$, show that $\lim_{\epsilon \to 0^+} f(x + \epsilon) = f(x)$ means that for every $\epsilon' > 0$, there is a $D > 0$, so that $[x, x + D) \subset f^{-1}(f(x) - \epsilon', f(x) + \epsilon')$.