MAT 145 : Homework Solutions

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Homework 1 Solutions

Problems from ***Review of Set Theory*** Notes

3 (c) We prove that $A \times (B - C) = (A \times B) - (A \times C)$. We show that $A \times (B - C) \subset (A \times B) - (A \times C)$, and $(A \times B) - (A \times C) \subset A \times (B - C)$. For the first containment, let $x \in A \times (B - C)$. Then $x = (a, b)$ where $a \in A$ and $b \in B - C$. We certainly have $x \in A \times B$, and since the second coordinate of x is not in C, we conclude that $x \notin A \times C$. For the second containment, let $x \in (A \times B) - (A \times C)$. Then $x = (a, b)$ where $a \in A$ and $b \in B$, but $x \notin A \times C$. Hence $b \notin C$. Therefore $x \in A \times (B - C)$.

3 (d) We prove that $(A - C) \times (B - D) \subset (A \times B) - (C \times D)$. Let $x \in (A - C) \times (B - D)$. Then $x = (a, b)$ where $a \in A - C$ and $b \in B - D$. We certainly have $x \in A \times B$. Since $a \notin C$, we conclude that $x = (a, b) \notin C \times D$. Therefore, $x \in (A \times B) - (C \times D)$. A counter-example for the reverse containment can be obtained by setting $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = B$. Then $(A \times B) - (C \times D) = \{(1, 2)\}$, while $(A - C) \times (B - D) = \varnothing$. Therefore, the reverse containment does not hold.

5 (e) Let $X \subset A$. To show that $X \subset f^{-1}(f(X))$, let $x \in X$ be given. Then $f(x) \in f(X)$, by definition of $f(X)$. By definition of pre-image of f, we may conclude that $x \in f^{-1}(f(X))$.

5 (f) Suppose that $f(f^{-1}(U)) = U$ for every $U \subset B$. To show that f is surjective, we show that $f(A) = B$. We set $U = B$ and use our hypothesis. Thus $f(f^{-1}(B)) = B$. Since A is the domain of f, we have $f^{-1}(B) = A$. Hence $f(A) = B$. Therefore, f is surjective.

Alternatively, we prove that f is surjective by showing that for each $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Let $b \in B$ be given. Set $U = \{b\}$; by hypothesis, $f(f^{-1}(\{b\})) = \{b\}$. This implies that $f^{-1}(\{b\}) \neq \varnothing$; so there is some element $a \in f^{-1}(\{b\})$. Then $f(a) = b$. Therefore f is surjective.

5 (g) Suppose that $f^{-1}(f(X)) = X$ for every $X \subset A$. To show that f is injective, we show that $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$. Let $a_1, a_2 \in A$ be given with $a_1 \neq a_2$. Set $X = \{a_1\}$. By hypothesis, $f^{-1}(f(\{a_1\})) = \{a_1\}$. Thus $a_2 \notin f^{-1}(f(\{a_1\}))$; so $f(a_2) \notin f(\{a_1\})$ by definition of pre-image of f. Hence $f(a_2) \neq f(a_1)$. Therefore f is injective.

Homework 2 Solutions

Problem from Crossley's Book

3.5 Let $\mathcal{T} = \{\varnothing\} \cup \{A \subset X : \text{ for every } a \in A, B_\delta(a) \subset A \text{ for some } \delta > 0\}$. We show that $\mathcal{T}$ is a topology on $\mathbb{R}^2$.

By definition of $\mathcal{T}$, $\varnothing \in \mathcal{T}$. Since each $a \in \mathbb{R}^2$ is contained in $B_1(a) \subset \mathbb{R}^2$, we conclude that $\mathbb{R}^2 \in \mathcal{T}$.

Let $\{A_\alpha\}_{\alpha \in J} \subset \mathcal{T}$. Set $A = \left(\bigcup_{\alpha \in J} A_\alpha\right)$; we show that $A \in \mathcal{T}$. Let $a \in A$. Then $a \in A_\alpha$ for some α. Since $A_\alpha \in \mathcal{T}$, there is a $\delta > 0$ such that $B_\delta(a) \subset A_\alpha \subset A$. This shows that $A \in \mathcal{T}$. Therefore, arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.

Let $A_1, A_2 \in \mathcal{T}$. We show that $A_1 \cap A_2 \in \mathcal{T}$. Let $a \in A_1 \cap A_2$. Then there exists $\delta_1, \delta_2 > 0$, such that $B_{\delta_i}(a) \subset A_i$ for $i = 1, 2$. Set $\delta = \min\{\delta_1, \delta_2\}$; it is easy to see that $B_\delta(a) \subset B_{\delta_i}(a)$ for each $i = 1, 2$. So $B_\delta(a) \subset A_1 \cap A_2$. Therefore $A_1 \cap A_2 \in \mathcal{T}$. A straightforward induction argument shows that finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.[1]

This completes the proof that $\mathcal{T}$ is a topology on $\mathbb{R}^2$.

Problems from Hatcher's Notes

5.(a) In order to prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we prove the containments

[1] As discussed in class, we do not have to give the induction argument. Showing that *the intersection of two open sets is open* suffices.

1. $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$, and
2. $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

For the first containment, let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. Thus every neighborhood of x intersects A, or every neighborhood of x intersects B. Since $A \subset A \cup B$ and $B \subset A \cup B$, we conclude that every neighborhood of x intersects $A \cup B$. Thus $x \in \overline{A \cup B}$.

For the second containment, it is convenient to proceed by contradiction. Assume that $x \in \overline{A \cup B}$, but $x \notin \overline{A} \cup \overline{B}$. Then $x \notin \overline{A}$ and $x \notin \overline{B}$. So x has neighborhoods O_A and O_B so that $O_A \cap A = O_B \cap B = \varnothing$. Therefore $O_A \cap O_B$ is a neighborhood of x with the property that $(O_A \cap O_B) \cap (A \cup B) = \varnothing$. This contradicts the assumption that $x \in \overline{A \cup B}$.

5.(d) We establish the containment $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$. Let $x \in \text{int}(A) \cup \text{int}(B)$. Then $x \in \text{int}(A)$ or $x \in \text{int}(B)$. So there exists a neighborhood O of x such that $x \in O \subset A$ or $x \in O \subset B$. In either case, O is a neighborhood of x such that $x \in O \subset A \cup B$. Therefore $x \in \text{int}(A \cup B)$.

To show that $\text{int}(A) \cup \text{int}(B) \neq \text{int}(A \cup B)$, we could set $A = (-\infty, 0]$ and $B = [0, \infty)$. Then $\text{int}(A) = (-\infty, 0)$ and $\text{int}(B) = (0, \infty)$. So $\text{int}(A) \cup \text{int}(B) = \mathbb{R} - \{0\}$, while $\text{int}(A \cup B) = \text{int}(\mathbb{R}) = \mathbb{R}$.

9. Let $\mathcal{T}_X$ denote the given topology on X and let $\mathcal{T}_Y$ denote the (induced) subspace topology on Y. We show that $\text{int}_X(A) \subset \text{int}_Y(A)$. Let $b \in \text{int}_X(A)$. Then there is a neighborhood $O_X \in \mathcal{T}_X$ of b such that $b \in O_X \subset A$. Since $O_X \subset A \subset Y$, we see that $O_X = O_X \cap Y$. So $O_X \in \mathcal{T}_Y$ by definition of subspace topology on Y. Therefore $b \in \text{int}_Y(A)$. This establishes that $\text{int}_X(A) \subset \text{int}_Y(A)$.

To show that $\text{int}_X(A) \neq \text{int}_Y(A)$, we could set $X = \mathbb{R}$, $Y = (-\infty, 0]$ and $A = (-1, 0]$. Then $\text{int}_Y(A) = A$, while $\text{int}_X(A) = (-1, 0) \neq A$.

14. In order to show that $f : X \to Y$ is continuous, we prove that the pre-image of each open set of Y is open in X. Let U be an open set in Y. By assumption, the restricted function $f_\alpha : O_\alpha \to Y$ is continuous for each α. Since $X = \bigcup O_\alpha$, we observe that if $x \in X$, then $x \in O_\alpha$ for some α.

An element-chasing argument shows that $f^{-1}(U) = \bigcup f_\alpha^{-1}(U)$:

$$x \in f^{-1}(U) \iff f(x) \in U \iff f_\alpha(x) \in U \text{ for some } \alpha \iff x \in f_\alpha^{-1}(U) \text{ for some } \alpha\,.$$

We can now finish the proof. Since each $f_\alpha^{-1}(U)$ is open in O_α, and O_α is an open set in X, we have that $f_\alpha^{-1}(U)$ is an open set in X by Problem 10 (in Chapter 1 of [Ha]).[2] So $f^{-1}(U)$ is a union of open sets in X; thus $f^{-1}(U)$ is open in X. Since U was arbitrary, we conclude that f is continuous.

Homework 3 Solutions

Problems from Crossley's Book

4.1 We proceed by contradiction. Assume that there exists a continuous function $f : [0,1] \to S$. Since U and V are open in S, the sets $f^{-1}(U)$ and $f^{-1}(V)$ are open in $[0,1]$. Also, $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$; so $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Since $S = U \cup V$, we have $[0,1] = f^{-1}(S) = f^{-1}(U) \cup f^{-1}(V)$. Thus $[0,1]$ is the union of two disjoint nonempty open subsets, implying that $[0,1]$ is disconnected; this is a contradiction.

4.8 Note that this result is trivial for the cases $n = 1$ and $n = 2$. We proceed by induction. Let $n \in \mathbb{N}$ with $n \geq 3$ be given.

($\star$) Assume that for every list of $n-1$ distinct points $x_1, \ldots, x_{n-1}$ in T, there are open sets $U_1, \ldots, U_{n-1}$ each containing one, and only one, of the points $x_1, \ldots, x_{n-1}$. We show that the same phenomenon occurs for any list of n distinct points.

Let $x_1, \ldots, x_n$ be distinct points in T. Since T is Hausdorff, for each $i = 1, \ldots, n-1$, there are neighborhoods V_i' and V_i'' of x_i and x_n, respectively, such that $V_i' \cap V_i'' = \varnothing$. Define $V_i = U_i \cap V_i'$ (U_i defined from the induction hypothesis ($\star$)) for each $i = 1, \ldots, n-1$; then define

$$V_n = \bigcap_{i=1}^{n-1} V_i'' \ .$$

We claim that $V_1, \ldots, V_n$ are open sets in T such that each V_i contains one, and only one of the points $x_1, \ldots, x_n$. By definition of our sets $V_1, \ldots, V_n$, the following is true for each $i = 1, \ldots, n-1$:

[2] Problem 10 (in Chapter 1 of [Ha]) was done in class, so it may be assumed for this problem.

- V_i is an intersection of open sets U_i and V_i', so V_i is open.
- V_n is the finite intersection of open sets, so V_n is open.
- $x_i \in V_i$, and $x_n \in V_n$.
- $x_i \notin V_n$, and $x_n \notin V_i$.
- $x_j \notin U_i$ for any $j \in \{1, \ldots, n-1\} - \{i\}$; hence $x_j \notin V_i$ for any such j.

Therefore, for each $i \in \{1, \ldots, n\}$, we have $x_i \in V_i$, V_i is open, and $x_j \notin V_j$ whenever $j \neq i$. This shows that each V_i contains one, and only one of the points $x_1, \ldots, x_n$.

4.10 Let L denote the real line with a double point at 0 (see page 52). We consider L as $\mathbb{R} \cup \{0'\}$. Then we may describe the (proposed) topology $\mathcal{T}_L$ of L as follows. Consider the function $p : L \to \mathbb{R}$ defined by $p(0') = 0$ and $p(t) = t$ for $t \neq 0'$; then $U \in \mathcal{T}_L$ if and only if $p(U) \in \mathcal{T}$. We now verify the four axioms for $\mathcal{T}_L$ to be a topology.

$\varnothing \in \mathcal{T}_L$ because $p(\varnothing) = \varnothing \in \mathcal{T}$.

$L \in \mathcal{T}_L$ because $p(L) = \mathbb{R} \in \mathcal{T}$.

As for arbitrary unions: Let $\{O_\alpha\} \subset \mathcal{T}_L$ be a collection of elements of $\mathcal{T}_L$. Set $O = \bigcup O_\alpha$. We show that $O \in \mathcal{T}_L$. Well, $p(O) = p(\bigcup O_\alpha) = \bigcup p(O_\alpha)$ from set theory. Since $p(O_\alpha) \in \mathcal{T}$ and $\mathcal{T}$ is a topology on $\mathbb{R}$, $p(O) = \bigcup p(O_\alpha) \in \mathcal{T}$. Thus $O \in \mathcal{T}_L$.

As for finite intersections: The following lemma will be useful.

Lemma. *For any two subsets $A, B \subset L$, we have*

$$p(A \cap B) = p(A) \cap p(B) \quad \text{or} \quad p(A \cap B) = (p(A) \cap p(B)) - \{0\} \ .$$

Proof of Lemma. First we show that $(p(A) \cap p(B)) - \{0\} \subset p(A \cap B)$. Let $t \in (p(A) \cap p(B)) - \{0\}$. Thus $t \in p(A) \cap p(B)$, $t \in \mathbb{R} - \{0\}$ and $p(t) = t$. The conditions $t \in p(A) \cap p(B)$ and $p(t) = t$ imply that $t \in A$ and $t \in B$; hence $t \in A \cap B$.[3] Since $p(t) = t$, we conclude that $t \in p(A \cap B)$.

From basic set theory, we always have $p(A \cap B) \subset p(A) \cap p(B)$.

[3] We are not saying that $p(A) = A$ or $p(B) = B$. We are concerned only with t.

There are now two cases: either $0 \notin p(A \cap B)$ or $0 \in p(A \cap B)$. If $0 \notin p(A \cap B)$, then $(p(A) \cap p(B)) - \{0\} = p(A \cap B) - \{0\} = p(A \cap B)$. If $0 \in p(A \cap B)$, then $\{0\} \cup ((p(A) \cap p(B)) - \{0\}) \subset p(A \cap B)$; therefore $p(A) \cap p(B) = p(A \cap B)$. $\square$

To finish the problem, we let $O_1, O_2 \in \mathcal{T}_L$. By the above lemma, we have $p(O_1 \cap O_2) = p(O_1) \cap p(O_2)$ or $p(O_1 \cap O_2) = (p(O_1) \cap p(O_2)) - \{0\} = p(O_1) \cap p(O_2) \cap (\mathbb{R} - \{0\})$. Since $p(O_1), p(O_2), \mathbb{R} - \{0\} \in \mathcal{T}$, we see that $p(O_1 \cap O_2)$ is an intersection of elements of $\mathcal{T}$. Thus $p(O_1 \cap O_2) \in \mathcal{T}$.

Problems from Hatcher's Notes

For Problem 15, let $\mathcal{T}$ denote the usual topology on $\mathbb{R}$, and let $\mathcal{T}_h$ denote the half-open interval topology on $\mathbb{R}$.

15(a) Let $A \subset \mathbb{R}$. We will prove that the following are equivalent.

(i) $x \in \overline{A}$ (with respect to $\mathcal{T}_h$).

(ii) There is a sequence $\{x_n\} \subset A$ such that $x_n \geq x$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} |x_n - x| = 0$.

First we show that (i) $\Rightarrow$ (ii). Let $x \in \overline{A}$ (with respect to $\mathcal{T}_h$). Then for each $n \in \mathbb{N}$, there exists $x_n \in [x, x + 1/n) \cap A$; note that $x_n \geq x$. Thus $0 \leq |x_n - x| \leq 1/n$ for all $x_n \in A$. By the "Squeeze Law"[4] (from Calculus), we conclude that $\lim_{n\to\infty} |x_n - x| = 0$, where $x_n \geq x$ and $x_n \in A$ for every $n \in \mathbb{N}$. Therefore (ii) holds.

We now show that (ii) $\Rightarrow$ (i). Suppose that $x \in \mathbb{R}$ and (ii) holds. So there is a sequence $\{x_n\} \subset A$ such that $x_n \geq x$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} |x_n - x| = 0$. Let U be a neighborhood of x; we will show that $U \cap A \neq \varnothing$. Since U is a neighborhood of x, there exists a basis element $[a, b) \in \mathcal{T}_h$ with $x \in [a, b) \subset U$. Since $x \neq b$, there is a $k \in \mathbb{N}$ such that $[x, x + 1/k) \subset [a, b)$. Since $\lim_{n\to\infty} |x_n - x| = 0$ where $x_n \geq x$ for all $n \in \mathbb{N}$, there is some $N \in \mathbb{N}$ for which $x_N \in [x, x + 1/k)$. So $x_N \in U \cap A$. Since U was an arbitrary neighborhood of x, we conclude that $x \in \overline{A}$ (with respect to $\mathcal{T}_h$). Therefore (i) holds.

15(b) Here, we let $\mathbb{R}_h$ denote the real line with the half-open interval topology. We will show that the following are equivalent

[4] also known as the "Sandwich Law"

(i) $f : \mathbb{R}_h \to \mathbb{R}$ is continuous.

(ii) $\lim_{\epsilon \to 0^+} f(x+\epsilon) = f(x)$ (i.e. f is continuous on the right) for every $x \in \mathbb{R}_h$.

First, we show that (i) $\Rightarrow$ (ii). Let $x \in \mathbb{R}_h$ be fixed. Let $\epsilon' > 0$ be given. We want to show that there exists a $D > 0$, so that $|f(x+\epsilon) - f(x)| < \epsilon'$ for every $0 < \epsilon < D$; this will establish (ii). By (i), the pre-image of the open set $(f(x) - \epsilon', f(x) + \epsilon') \in \mathcal{T}$ is open in $\mathcal{T}_h$. So there is a basis element $[a, b) \in \mathcal{T}_h$ such that $x \in [a, b) \subset f^{-1}(f(x) - \epsilon', f(x) + \epsilon')$. Set $D = b - x$. Then $[x, x+D) = [x, b) \subset [a, b) \subset f^{-1}(f(x) - \epsilon', f(x) + \epsilon')$. Note that for every $0 < \epsilon < D$, we have $x + \epsilon \in [x, x+D) = [x, b)$; it is now clear that $|f(x+\epsilon) - f(x)| < \epsilon'$ for every $0 < \epsilon < D$. So (ii) holds.

We now show that (ii) $\Rightarrow$ (i). Suppose that (ii) holds for each $x \in \mathbb{R}_h$. Let $U \in \mathcal{T}$; we will show that $f^{-1}(U) \in \mathcal{T}_h$. We may as well assume that $f^{-1}(U) \neq \varnothing$. For each $x \in f^{-1}(U)$, consider $f(x)$. Since $U \in \mathcal{T}$, there exists $\epsilon' > 0$ such that $(f(x) - \epsilon', f(x) + \epsilon') \subset U$. By (ii), there exists a $D > 0$, so that $|f(x+\epsilon) - f(x)| < \epsilon'$ for every $0 < \epsilon < D$; in other words, $[x, x+D) \subset f^{-1}(f(x) - \epsilon', f(x) + \epsilon') \subset f^{-1}(U)$. Set $N_x = [x, x + D/2)$. Then $N_x \subset f^{-1}(U)$ and $N_x \in \mathcal{T}_h$. We define N_x for every $x \in \mathbb{R}_h$ in this way. Therefore

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} N_x$$

is a union of elements of $\mathcal{T}_h$. Thus $f^{-1}(U) \in \mathcal{T}_h$. So (i) holds.

Homework 4 Solutions

Problems from Crossley's Book

5.1 We construct a homeomorphism $f : [1, 2) \to (-1, 0]$. Construct the line in $\mathbb{R}^2$ passing through the points $(1, 0)$ and $(2, -1)$; this will contain the graph of $f(x)$. We calculate an equation for this line as $y = -x+1$. So define $f(x) = -x+1$. This clearly defines a continuous function; the only thing to check is that $f([1, 2)) \subset (-1, 0]$. Indeed, basic algebra shows that $1 \leq x < 2$ implies $-1 < -x + 1 \leq 0$. Solving the equation $y = -x + 1$ for x produces the inverse function $g : (-1, 0] \to [1, 2)$ defined by $g(y) = 1 - y$. It is straightforward to see that g is continuous,

$(g \circ f)(x) = x$ for all $x \in [1,2)$, and $(f \circ g)(y) = y$ for all $y \in (-1,0]$. This shows that f is a homeomorphism. Therefore, the intervals $[1,2)$ and $(-1,0]$ are homeomorphic.

5.5 For stereographic projection of $\mathbb{S}^1$: Consider $\mathbb{R}^2$ as $\{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$. Let (x,y) be a point in $\mathbb{S}^1 - \{(0,1)\}$. An equation for the line passing through $(0,1)$ and (x,y) is

$$x_2 - 1 = \left(\frac{y-1}{x}\right) x_1 \ .$$

The intersection point of this line with the projection line $\{(x_1, -1) : x_1 \in \mathbb{R}\}$ is obtained by solving

$$-1 - 1 = \left(\frac{y-1}{x}\right) x_1$$

for x_1; hence $\big(2x/(1-y), -1\big)$ is the intersection point. Therefore, stereographic projection $f : \mathbb{S}^1 - \{(0,1)\} \to \mathbb{R}$ (as in Example 5.7) is given by the formula $f(x,y) = 2x/(1-y)$.

For stereographic projection of $\mathbb{S}^2$: Consider $\mathbb{R}^3$ as $\{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$. Let (x,y,z) be a point in $\mathbb{S}^2 - \{(0,0,1)\}$. The straight line containing $(0,0,1)$ and (x,y,z) can be parametrized by

$$\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3 \ , \ \boldsymbol{r}(t) = (1-t)\big(0,0,1\big) + t\big(x,y,z\big) \ .$$

Associated parametric equations for this line are

$$x_1 = tx \ , \ x_2 = ty \ , \ x_3 = (1-t) + tz \ .$$

To find the intersection of this line with the projection plane $\{(x_1, x_2, -1) : x_1, x_2 \in \mathbb{R}\}$, we solve $x_3 = -1$ for t, then use this t-value to find the intersection point. We see that $x_3 = -1$ has solution $t = 2/(1-z)$. So the intersection point is $\big(2x/(1-z), 2y/(1-z), -1\big)$. Therefore, stereographic projection $f : \mathbb{S}^2 - \{(0,0,1)\} \to \mathbb{R}^2$ (as in Example 5.7) is given by the formula $f(x,y,z) = \big(2x/(1-z), 2y/(1-z)\big)$.

Problems from Hatcher's Notes

3. Let X be the real-line with the finite complement topology. We show that X is compact. Let $\{O_\alpha\}$ be an open covering of X; we find a finite subcovering. Let O_{α_0} be a nonempty element of the covering. Then $O_{\alpha_0} = X - \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n \in X$. Since $\{O_\alpha\}$ is a covering of X, there exists elements $O_{\alpha_1}, \ldots, O_{\alpha_n}$ of the covering such that $x_i \in O_{\alpha_i}$ for each $i = 1, \ldots, n$. It is now clear that $\{O_{\alpha_0}, O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ is a finite subcovering of $\{O_\alpha\}$. It follows that X is compact.

5. Let $x_1, x_2 \in X$ be distinct points; we show that there are disjoint neighborhoods of x_1 and x_2, respectively, in X. Consider $f(x_1), f(x_2) \in Y$. Since f is injective, the values $f(x_1)$ and $f(x_2)$ are distinct. Since Y is Hausdorff, there are disjoint neighborhoods V_1 and V_2 of $f(x_1)$ and $f(x_2)$, respectively, in Y. Since f is continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are neighborhoods of x_1 and x_2, respectively, in X. Furthermore, by definition of pre-image and the fact that $V_1 \cap V_2 = \varnothing$, it is clear that $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. This establishes that X is Hausdorff.

6. We consider A, B, $A \cup B$ and $A \cap B$ to be subspaces of X. We also use that fact that the subspace topology that $A \cap B$ inherits from $A \cup B$ is the same topology that $A \cap B$ inherits from X.

   First we show that $A \cup B$ is compact. Let $\{O_\alpha\}$ be an open covering of $A \cup B$. Then $\{O_\alpha \cap A\}$ (resp. $\{O_\beta \cap B\}$) is an open covering of A (resp. B). By compactness of A (resp. B), there exists a subcovering $\{O_{\alpha_1} \cap A, \ldots, O_{\alpha_m} \cap A\}$ (resp. $\{O_{\beta_1} \cap B, \ldots, O_{\beta_n} \cap B\}$) of A (resp. B). It follows that

   $$\{O_{\alpha_1}, \ldots, O_{\alpha_m}, O_{\beta_1}, \ldots, O_{\beta_n}\}$$

   forms a subcovering of $\{O_\alpha\}$ for $A \cup B$.

   Now assume that X is a Hausdorff space, so A and B are Hausdorff subspaces; we show that $A \cap B$ is compact. Since A and B are compact subspaces of the Hausdorff space X, we conclude that A and B are closed in X.[5] Thus the finite intersection $A \cap B$ is closed in X; hence $A \cap B$ is closed in A.[6] Since $A \cap B$ is a closed subspace of the compact space A, we conclude that $A \cap B$ is a compact subspace of A.[7] Since the subspace topology that $A \cap B$ inherits from A is the

[5]This follows from the proposition on page 35 of Hatcher's notes.
[6]This follows from the lemma on page 11 of Hatcher's notes.
[7]This follows from a proposition on page 32 of Hatcher's notes.

same topology that $A \cap B$ inherits from X; it follows that $A \cap B$ is a compact subspace of X.

Homework 5 Solutions

Problems from Crossley's Book

5.9 Suppose that S and T are *Hausdorff* spaces; we show that the product $S \times T$ is a *Hausdorff* space. Let $(s_1, t_1), (s_2, t_2) \in S \times T$ be distinct points. Then $s_1 \neq s_2$ or $t_1 \neq t_2$. Let's assume that $s_1 \neq s_2$; the proof under the assumption $t_1 \neq t_2$ is similar. Since $s_1 \neq s_2$ and S is Hausdorff, there exists disjoint neighborhoods $U_1 \subset S$ and $U_2 \subset S$ of s_1 and s_2 respectively. Set $V_1 = U_1 \times T$ and $V_2 = U_2 \times T$. Then V_1 and V_2 are disjoint neighborhoods of (s_1, t_1) and (s_2, t_2) respectively.

Let S and T be spaces, and let $S \times T$ be the product space. Suppose that $S \times T$ is *connected*; we show that S and T are both *connected*:

Solution 1: Let $p_S : S \times T \to S$ and $p_T : S \times T \to T$ be the associated projection functions; in class it was established that these functions are continuous and surjective. Then S is the image of a connected space under the continuous function; therefore, S is connected. Similarly, we can prove that T is connected.

Solution 2: We proceed by contradiction. Suppose that $S = A \cup B$ were a separation of S. Then A and B are disjoint nonempty open sets in S. Consequently, $A \times T$ and $B \times T$ are disjoint nonempty open sets in $S \times T$: For any $a \in A$ and $t \in T$, we have $(a, t) \in A \times T$; so $A \times T \neq \varnothing$. For any $b \in B$ and $t \in T$, we have $(b, t) \in B \times T$; so $B \times T \neq \varnothing$. The sets $A \times T$ and $B \times T$ are basis elements for the product topology on $S \times T$, so they are open. Lastly, $(A \times T) \cap (B \times T) = (A \cap B) \times T = \varnothing \times T = \varnothing$; hence $A \times T$ and $B \times T$ are disjoint. Since $S = A \cup B$, it is straightforward to see that $S \times T = (A \times T) \cup (B \times T)$. Therefore $S \times T = (A \times T) \cup (B \times T)$ is a separation of $S \times T$, a contradiction to the assumption that $S \times T$ is connnected. Similarly, we can prove that T is connected.

Let S and T be spaces, and let $S \times T$ be the product space. Suppose that $S \times T$ is *compact*; we show that S and T are both *compact*.

Solution 1: Let $p_S : S \times T \to S$ and $p_T : S \times T \to T$ be the associated projection functions; in class it was established that these functions are continuous and surjective. Then S is the image of a compact space under the continuous function; therefore, S is compact. Similarly, we can prove that T is compact.

Solution 2: Let $\{O_\alpha\}$ be an open covering of S; we show that there is a finite subcovering. Since each O_α is open in S, the subset $O_\alpha \times T$ is open in $S \times T$. It follows that $\{O_\alpha \times T\}$ is an open covering of $S \times T$. By compactness of $S \times T$, there is a finite subcovering $\{O_{\alpha_1} \times T, \ldots, O_{\alpha_n} \times T\}$ of $S \times T$. Consequently, the collection $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ forms a subcovering of S. Therefore S is compact. Similarly, we can prove that T is compact.

Problems from Hatcher's Notes

7. Let $A \subset X$ and $B \subset Y$ be subspaces. We show that $\overline{A \times B} = \overline{A} \times \overline{B}$ and $\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$. The equality $\overline{A \times B} = \overline{A} \times \overline{B}$ follows from

$$\begin{aligned} (x,y) \in \overline{A \times B} &\iff \text{every basis-element neighborhood } U \times V \text{ of } (x,y) \text{ intersects } A \times B \\ &\iff \text{every neighborhood } U \text{ of } x \text{ intersects } A, \text{ and} \\ &\qquad \text{every neighborhood } V \text{ of } y \text{ intersects } B \\ &\iff x \in \overline{A} \text{ and } y \in \overline{B} \\ &\iff (x,y) \in \overline{A} \times \overline{B}\ . \end{aligned}$$

The equality $\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$ follows from

$$\begin{aligned} (x,y) \in \operatorname{int}(A \times B) &\iff \text{there is a basis-element neighborhood } U \times V \subset A \times B \text{ of } (x,y) \\ &\iff \text{there is a neighborhood } U \text{ of } x \text{ contained in } A, \text{ and} \\ &\qquad \text{there is a neighborhood } V \text{ of } y \text{ contained in } B \\ &\iff x \in \operatorname{int}(A) \text{ and } y \in \operatorname{int}(B) \\ &\iff (x,y) \in \operatorname{int}(A) \times \operatorname{int}(B)\ . \end{aligned}$$

14. To show that the function $d : X \times X \to \mathbb{R}$ is continuous, we show that $d^{-1}(\alpha, \beta)$ is open for every basis element $(\alpha, \beta) \subset \mathbb{R}$. So let (α, β) be an open interval in $\mathbb{R}$. We

may as well assume that $d^{-1}(\alpha, \beta) \neq \varnothing$; so $\beta > 0$. Now let $(x, x') \in d^{-1}(\alpha, \beta)$; we show that (x, x') is an interior point. It will be convenient to set $a = d(x, x')$. Since $\alpha < a < \beta$, there exists a number $r > 0$ for which $\alpha < a - 2r < a < a + 2r < \beta$.

Define $V = B_r(x)$ and $V' = B_r(x')$; then $V \times V'$ is a neighborhood of (x, x') in $X \times X$. We show that $V \times V' \subset d^{-1}(\alpha, \beta)$ by an element-chasing argument. Let $(y, y') \in V \times V'$; we will show that $\alpha < d(y, y') < \beta$.

To see that $d(y, y') < \beta$, we use the triangle inequality:

$$\begin{aligned} d(y, y') &\leq d(y, x) + d(x, y') \\ &\leq d(y, x) + d(x, x') + d(x', y') \\ &< r + a + r \\ &= a + 2r \\ &< \beta\ . \end{aligned}$$

We similarly show that $\alpha < d(y, y')$:

$$\begin{aligned} a = d(x, x') &\leq d(x, y) + d(y, x') \\ &\leq d(x, y) + d(y, y') + d(y', x') \\ &< r + d(y, y') + r \\ &= d(y, y') + 2r\ . \end{aligned}$$

Therefore $a < d(y, y') + 2r$. Subtracting $2r$ from both sides of this inequality yields $a - 2r < d(y, y')$; hence $\alpha < d(y, y')$. This establishes that $V \times V' \subset d^{-1}(\alpha, \beta)$. So (x, x') is an interior point of $d^{-1}(\alpha, \beta)$. Since (x, x') was an arbitrary point of $d^{-1}(\alpha, \beta)$, we conclude that $d^{-1}(\alpha, \beta)$ is open in $X \times X$. Therefore, $d : X \times X \to \mathbb{R}$ is a continuous function.