MAT 145 : Quiz Solutions

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Quiz 1 Solutions

1. Let the set $X = \{a, b, c, d\}$ be given the topology $\mathcal{T} = \{\varnothing, X, \{c\}, \{d\}, \{c, d\}, \{a, b, c\}\}$. Let S be the subset $S = \{a, c, d\} \subset X$.

   (a) List the elements of the subspace topology $\mathcal{T}_S$ on S.

   Solution: By definition of $\mathcal{T}_S$, we have $\mathcal{T}_S = \{O \cap S : O \in \mathcal{T}\}$. Therefore $\mathcal{T}_S = \{\varnothing, S, \{c\}, \{d\}, \{c, d\}, \{a, c\}\}$.

   (b) With respect to the topologies $\mathcal{T}$ on X and $\mathcal{T}_S$ on S, determine whether or not the function

   $$f : S \to X \ , \ f(a) = a \ , f(c) = d \ , f(d) = c$$

   is continuous. Justify your answer.

   Solution: The function f is not continuous: for f to be continuous, the pre-image of any open set of X has to be an open set of S; in other words, $f^{-1}(O) \in \mathcal{T}_S$ whenever $O \in \mathcal{T}$. By considering the pre-image of each point in $\{a, b, c\}$, we see that

   $$f^{-1}(\{a, b, c\}) = \{a, d\} \ .$$

   Since $\{a, b, c\} \in \mathcal{T}$ while $f^{-1}(\{a, b, c\}) = \{a, d\} \notin \mathcal{T}_S$, f cannot be continuous.

Quiz 2 Solutions

1. Let X be a space. For any subset $A \subset X$, prove that

   $$\partial A = \varnothing \iff A \text{ is both open and closed in } X \ .$$

Elementary Solution: Recall the definition of ∂A:

$$\partial A = \{x \in X : \text{every neighborhood of } x \text{ intersects } A \text{ and } X - A\} .$$

Also recall that A is closed if and only if $X - A$ is open. Suppose that $\partial A = \varnothing$. Given $x \in X$, there exists a neighborhood O_A of x that does not intersect $X - A$ (hence $x \in O_A \subset A$), or there exists a neighborhood O_{X-A} of x that does not intersect A (hence $x \in O_{X-A} \subset X - A$). In particular, for every $x \in A$, there exists a neighborhood O_A of x such that $x \in O_A \subset A$, so A is open; similarly, for every $x \in X - A$, there exists a neighborhood O_{X-A} of x such that $x \in O_{X-A} \subset X - A$, so $X - A$ is open. This establishes that $\partial A = \varnothing$ implies that A is open and closed.

Now suppose that A is open and closed; hence A and $X - A$ are open. Let $x \in X$. We show that $x \notin \partial A$. If $x \in A$, then A *itself* is a neighborhood x that does not intersect $X - A$ (because A is open). If $x \in X - A$, then $X - A$ *itself* is a neighborhood of x that does not intersect A (because $X - A$ is open). In either case ($x \in A$ or $x \in X - A$), we have that $x \notin \partial A$. This establishes that if A is open and closed, then $\partial A = \varnothing$.

Nonelementary Solution: From class and Hatcher's notes, we may use the facts

(1) $\text{int}(A) \cup \partial A = \bar{A}$.

(2) $\text{int}(A) \subset A \subset \bar{A}$.

(3) A is open if and only if $\text{int}(A) = A$; A is closed if and only if $A = \bar{A}$.

By (1), we immediately have $\partial A = \varnothing$ if and only if $\text{int}(A) = \bar{A}$. By combining this with (2), we see that $\partial A = \varnothing$ if and only if $\text{int}(A) = A = \bar{A}$. Therefore, by (3), $\partial A = \varnothing$ if and only if A is open ($\text{int}(A) = A$) and closed ($A = \bar{A}$).

2. Let X be a Hausdorff space, and let A be a subspace of X. Prove that A is a Hausdorff space.

Solution: Let X be a Hausdorff space, and let A be a subspace of X. For clarity, we let $\mathcal{T}_X$ denote the given topology on X, and let $\mathcal{T}_A$ be the induced subspace topology on A. Let x_1 and x_2 be distinct points in A; we will show that there exists disjoint neighborhoods (from $\mathcal{T}_A$) of x_1 and x_2. Since X is Hausdorff, there exists a neighborhood $O_1' \in \mathcal{T}_X$ of x_1 and there exists a neighborhood $O_2' \in \mathcal{T}_X$ of x_2 such that $O_1' \cap O_2' = \varnothing$. Define $O_1 = O_1' \cap A \in \mathcal{T}_A$, and define $O_2 = O_2' \cap A \in \mathcal{T}_A$. By definition of $\mathcal{T}_A$, we see that O_1 is a neighborhood of x_1 (open in A) and O_2 is a neighborhood of x_2 (open in

A). We also see that

$$O_1 \cap O_2 = (O_1' \cap A) \cap (O_2' \cap A) = (O_1' \cap O_2') \cap A = \varnothing \ ,$$

since $O_1' \cap O_2' = \varnothing$. This establishes that the subspace A is a Hausdorff space.

3. Let the set $X = \{a, b, c, d\}$ be given the topology

$$\mathcal{T} = \{\varnothing, X, \{c\}, \{a, b, c\}\} \ .$$

(a) Prove or disprove: X is Hausdorff.

<u>Solution:</u> We prove that X is not Hausdorff. Observe that the only neighborhood of d is the whole space X because $d \notin \{c\}$ and $d \notin \{a, b, c\}$. Also observe that the only neighborhoods of b are X and $\{a, b, c\}$. So d and b do not possess disjoint neighborhoods. Therefore X is not Hausdorff.

(b) Prove directly that X is connected.

<u>Solution:</u> Suppose that $X = A \cup B$ were a separation of X (i.e. A and B are disjoint nonempty open sets whose union is X); we derive a contradiction. Observe that the only neighborhood of d is the whole space X. Since d lies in either A or B, we deduce that either $A = X$ or $B = X$; so, either $B = \varnothing$ or $A = \varnothing$. This contradicts that $X = A \cup B$ is a separation.

(c) Show that the subspace $S = \{a, b\}$ is path connected by explicitly defining a path between a and b.

<u>Solution:</u> Define a function $f : [0, 1] \to S$ by $f(t) = a$ for all $0 \le t \le 1/2$, and $f(t) = b$ for all $1/2 < t \le 1$. Note that the subspace topology on S is the indiscrete topology. It is easy to see that f is continuous (indeed, $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(S) = [0, 1]$ are open in $[0, 1]$). Therefore, f is a path in S from a to b.

(d) For each pair or points in X, explicitly define a path between these points. Deduce that X is actually path connected.

<u>Solution:</u> *This is similar to the the construction in part (c), but some care has to be taken when constructing paths involving the point c :*

To construct a path from a to c, define a function $f : [0, 1] \to X$ by $f(t) = a$ for all $0 \le t \le 1/2$, and $f(t) = c$ for all $1/2 < t \le 1$. All the relevant pre-images $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(X) = [0, 1]$, $f^{-1}(\{c\}) = (1/2, 1]$, and $f^{-1}(\{a, b, c\}) = [0, 1]$ are all open in $[0, 1]$. So f is a (continuous) path.

To construct a path from b to c, define a function $f : [0,1] \to X$ by $f(t) = b$ for all $0 \le t \le 1/2$, and $f(t) = c$ for all $1/2 < t \le 1$. All the relevant pre-images $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(X) = [0,1]$, $f^{-1}(\{c\}) = (1/2,1]$, and $f^{-1}(\{a,b,c\}) = [0,1]$ are all open in $[0,1]$. So f is a (continuous) path.

To construct a path from d to c, define a function $f : [0,1] \to X$ by $f(t) = d$ for all $0 \le t \le 1/2$, and $f(t) = c$ for all $1/2 < t \le 1$. All the relevant pre-images $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(X) = [0,1]$, $f^{-1}(\{c\}) = (1/2,1]$, and $f^{-1}(\{a,b,c\}) = (1/2,1]$ are all open in $[0,1]$. So f is a (continuous) path.

To construct a path from d to a, define a function $f : [0,1] \to X$ by $f(t) = d$ for all $0 \le t \le 1/2$, and $f(t) = a$ for all $1/2 < t \le 1$. All the relevant pre-images $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(X) = [0,1]$, $f^{-1}(\{c\}) = \varnothing$, and $f^{-1}(\{a,b,c\}) = (1/2,1]$ are all open in $[0,1]$. So f is a (continuous) path.

A path from a to b was already constructed in part (c). We can use the same formula for f. All the relevant pre-images $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(X) = [0,1]$, $f^{-1}(\{c\}) = \varnothing$, and $f^{-1}(\{a,b,c\}) = [0,1]$ are all open in $[0,1]$. So f is a (continuous) path.

Since we were able to construct all of the required paths in X, we deduce that X is path connected.

Quiz 3 Solutions

1. Give a self-contained proof of the following: Let X be a path connected space, and let Y be a space. Suppose that $f : X \to Y$ is a surjective continuous function. Show that Y is path connected.

   Solution: Let $y_1, y_2 \in Y$; we show that there exists a path in Y from y_1 to y_2. Since f is surjective, there exists $x_1, x_2 \in X$ for which $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is path connected, there exists a continuous function $g : [0,1] \to X$ for which $g(0) = x_1$ and $g(1) = x_2$. Since f and g are continuous, the composition $(f \circ g) : [0,1] \to Y$ is continuous. Furthermore, $(f \circ g)(0) = f(x_1) = y_1$ and $(f \circ g)(1) = f(x_2) = y_2$. Therefore, a path in Y from y_1 to y_2 exists. This establishes that Y is path connected.

2. Let X be a compact Hausdorff space, and let A be a closed subset of X. Suppose that $y \in X - A$. Prove that there exist open sets V and V' in X such that $y \in V$, $A \subset V'$, and $V \cap V' = \varnothing$.

Solution: Given $a \in A$, there exists disjoint neighborhoods V_a and V'_a of y and a respectively in X, since X is Hausdorff. We see that $\{V'_a \cap A\}_{a \in A}$ forms an open covering of A. Since A is closed in the compact space X, we have that A is compact. So there exists a finite subcovering $\{V'_{a_1} \cap A, \dots, V'_{a_n} \cap A\}$ of A. Set $V = \bigcap_{i=1}^{n} V_{a_i}$ and $V' = \bigcup_{i=1}^{n} V'_{a_i}$. Since $y \in V_{a_i}$ for each $i = 1, \dots, n$, we see that $y \in V$; since $V_{a_i} \cap V'_{a_i} = \varnothing$ for each $i = 1, \dots, n$, we see that $V \cap V' = \varnothing$. Since V is the finite intersection of open sets in X, we see that V is open is X. Since V' is the union of open sets of X, we see that V' is open in X. Finally, we see that $A \subset \bigcup_{i=1}^{n} V'_{a_i} = V'$. Therefore, there exist open sets V and V' in X such that $y \in V$, $A \subset V'$, and $V \cap V' = \varnothing$.