LEAVES WITHOUT HOLONOMY

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We prove the following result.

THEOREM. *Let M be a paracompact manifold with a foliation of codimension k. Let T be the union of all leaves with trivial holonomy. Then T is a dense G_δ in M.*

This result is also due independently to G. Hector [3], who has shown how useful it can be in understanding the geometry of certain foliated manifolds. In such applications one sometimes needs a form of this theorem which applies to foliated subspaces, for example a minimal subset of a foliation. In fact our proof goes through unaltered in the situation where M is a locally compact, paracompact, Hausdorff foliated space such that each plaque is locally connected. We do not need to assume that M is a manifold. (We recall that locally compact Hausdorff spaces satisfy the Baire category theorem.) Our treatment of the result differs from that of Hector in several respects. Firstly we give complete details of the proof. Secondly we allow the manifold which is foliated to be non-compact. Thirdly we do not restrict the differentiability class of the foliation.

Later we will give an example to show that T may be empty if M is not paracompact. We note that if M is a paracompact manifold, then the interior of T may e empty, and we will give an example which displays this behaviour.

Proof of the theorem. Let I be an indexing set for a family of admissible charts $h_i : P_i \times Q_i \to M$ $(i \in I)$. Let U_i be the image of h_i. We suppose that U_i is an open subset of M. In the case of a foliated manifold we suppose that P_i is an open disk in R^{n-k} and Q_i is an open disk in R^k. In the more general situation of a foliated space, we suppose only that P_i is a connected and open subspace of some locally compact, locally connected, Hausdorff space P and that Q_i is an open subspace of some locally compact Hausdorff space Q.

The foliation condition is that given $i, r \in I$ we have (locally) maps f_{ir} and g_{ir} such that g_{ir} is one-to-one and

$$h_i^{-1} h_r(x, y) = (f_{ir}(x, y), g_{ir}(y))$$

where $(x, y) \in h_r^{-1}(U_i \cap U_r) = h_r^{-1} U_i$. The maps g_{ir} are used to define the holonomy. For our purposes the word "locally" in the above definition is an embarrassment. This word means that given $(x_0, y_0) \in h_r^{-1} U_i$, there exists a small neighbourhood of (x_0, y_0) and maps f_{ir} and g_{ir} defined on this neighbourhood, such that the above formula holds on the neighbourhood. Therefore there may correspond to a fixed pair $i, r \in I$ many different holonomy maps g_{ir}.

To see the difficulty more clearly we give an example where $U_i \subset U_r$. Let R^3 be

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foliated by vertical lines. Let A be a smoothly embedded 2-dimensional ribbon, transverse to the foliation and almost horizontal, but such that projection onto a horizontal plane is not $1-1$ on A. For example, the projection of A could be a thickened figure of eight in the plane. If we now thicken A in the vertical direction, we obtain an admissible chart in an obvious way. But there is no single holonomy map going from an open subset of R^2 to A.

This difficulty is dealt with in the following technical lemma, which appeared implicitly in [1]. In view of the fact that it is the kind of technical result which is often needed in foliation theory, it seems worthwhile to give complete details here.

LEMMA. *Let M be a Hausdorff, locally compact, paracompact, foliated space with locally connected plaques. Let $h_i : P_i \times Q_i \to M$ $(i \in I)$ be a family of charts covering M such that each P_i is connected. Then there is a family of charts*

$$h_j : P_j \times Q_j \to M \quad (j \in J, I \cap J = \varnothing)$$

and a map $\theta : J \to I$ with the following properties:

(i) *P_j is connected.*

(ii) *$\bar{P}_j$ is a compact subspace of $P_{\theta j}$ and $\bar{Q}_j$ is a compact subspace of $Q_{\theta j}$.*

(iii) *h_j is the restriction of $h_{\theta j}$.*

(iv) *Writing $V_j = \operatorname{im} h_j$ and $U_i = \operatorname{im} h_i$, we have $\{V_j\}_{j \in J}$ is a star refinement of $\{U_i\}_{i \in I}$. That is to say, if $V_j \cap V_k \neq \varnothing$, then $\bar{V}_j \subset U_{\theta k}$.*

(v) *$\{V_j\}_{j \in J}$ is a locally finite family and each $\bar{V}_j$ is compact.*

(vi) *For $j, k \in J$, let $A_{jk} = \pi_2 h_j^{-1}(V_j \cap V_k) \subset Q_j$. Then there exists a unique map $g_{jk} : A_{kj} \to A_{jk}$ such that $\pi_2 h_j^{-1} h_k = g_{jk} \pi_2$ on $h_k^{-1}(V_j \cap V_k)$. Further, g_{jk} is a homeomorphism onto.*

In fact we will show that if a family of charts $\{h_j : j \in J\}$ satisfies (i)–(v) above, then (vi) is automatically satisfied.

Proof. Let $\{W_k\}_{k \in K}$ be a star-refinement of $\{U_i\}_{i \in I}$, associated with a mapping $\psi : K \to I$. We may assume that $\{W_k\}$ is locally finite and that each $\bar{W}_k$ is compact. We shrink the covering $\{W_k\}_{k \in K}$ to an open covering $\{W_k'\}_{k \in K}$ such that $\bar{W}_k' \subset W_k$ for each $k \in K$. For each $k \in K$, we now take a finite covering of $\bar{W}_k'$ by charts

$$h_{k,r} : P_{k,r} \times Q_{k,r} \to W_k,$$

where $P_{k,r}$ is a connected open subspace of $P_{\psi k}$, $Q_{k,r}$ is an open subspace of $Q_{\psi k}$ and $h_{k,r}$ is the restriction of $h_{\psi k}$. Clearly this gives us (i)–(v).

Now let $j, k \in J$ and let $y \in A_{jk} = \pi_2 h_j^{-1}(V_j \cap V_k)$. Then $h_j(P_j \times y) \cap V_k \neq \varnothing$. It follows that $h_j(P_j \times y) \subset U_{\theta k}$. Since P_j is connected and the inverse image under h_j of a plaque of $U_{\theta k}$ is an open subset of $P_j \times y$ by the foliation condition, we see that $h_j(P_j \times y)$ lies in a single plaque of $U_{\theta k}$. We can therefore write

$$g_{kj}(y) = \pi_2 h_{\theta k}^{-1} h_j(P_j \times y) \quad \text{for} \quad y \in A_{jk}.$$

By condition (iii) of the lemma, we have

$$g_{kj} \pi_2 = \pi_2 h_k^{-1} h_j \quad \text{on} \quad h_j^{-1}(V_j \cap V_k).$$

We can write this in the form $h_k^{-1} h_j(x, y) = (x', g_{kj} y)$, where x' depends on x and y. It follows immediately that g_{kj} is continuous.

We show that g_{kj} is a homeomorphism by proving that g_{jk} is its inverse. If $y \in A_{jk}$, there exists x such that $(x, y) \in h_j^{-1}(V_j \cap V_k)$. Then

$$\begin{aligned}(x, y) = (h_j^{-1} h_k)(h_k^{-1} h_j)(x, y) &= (h_j^{-1} h_k)(x', g_{kj} y)\\ &= (x'', g_{jk} g_{kj} y).\end{aligned}$$

This completes the proof of the lemma.

Now we fix a covering by admissible charts as given by the lemma. We define an equivalence relation on M by saying that z is equivalent to w if there is a finite chain $V_{j(1)}, \ldots, V_{j(n)}$ with $z \in V_{j(1)}$, $w \in V_{j(n)}$ and $V_{j(i-1)} \cap V_{j(i)} \neq \varnothing$. Each equivalence class is both open and closed. So there is no loss of generality in assuming that we have only one equivalence class. Since each V_i only meets a finite number of other sets V_j $(i, j \in J)$, we see that the indexing set J is countable.

Let c be a periodic function of the integers into J (that is, for some integer $n > 0$, $c(i+n) = c(i)$), such that $V_{c(i)} \cap V_{c(i+1)} \neq \varnothing$ for any i. There are at most countably many such functions c. Any such c gives rise to a holonomy map

$$g_c = g_{c(0)c(1)} g_{c(1)c(2)} \cdots g_{c(n-1)c(n)}.$$

As usual, composition of maps, where domain and range do not match, is defined by restricting to the largest possible domain and range so that they do match.

Let $z \in V_j \subseteq M$ and let $h_j : P_j \times Q_j \to V_j$ be the corresponding admissible chart. Let $q = \pi_2 h_j^{-1} z$. One way to define the holonomy group of the leaf through z at the point z is as follows. We take the group of germs of homeomorphisms from a neighbourhood of q in Q_j to a neighbourhood of q in Q_j, induced by g_c for some c as above, where $c(0) = j$ and $g_c(q) = q$.

Let the domain of g_c be D_c and let its range be R_c. Then D_c and R_c are open subsets of Q. Let F_c be the fixed point set of g_c. That is,

$$F_c = \{x \in Q : x \in D_c \cap R_c \quad \text{and} \quad g_c(x) = x\}.$$

F_c is closed in D_c and also in R_c. Let B_c be boundary of F_c in Q. Then B_c has void interior. Let B be the union of all the B_c (note that this is a countable union).

The next observation is that if $y \in Q_j \setminus B$ and $x \in P_j$ then $h_j(x, y)$ lies on a leaf with trivial holonomy. Suppose not. Choose a loop on the leaf along which the holonomy is not trivial. Let $V_{c(0)}, V_{c(1)}, \ldots, V_{c(n)}$ be any chain covering this loop such that $c(n) = c(0)$ and such that the loop can be cut up into n intervals, with the ith interval lying in $V_{c(i)}$. Then the holonomy along the loop is given by g_c. Moreover, the initial point of the loop corresponds to a fixed point y_0 of g_c. Now y_0 is not in the boundary of the fixed point set of g_c. Therefore y_0 has a neighbourhood N in Q which is entirely contained in the fixed point set of g_c. But then g_c is fixed on N so that the holonomy induced at y_0 by the loop is trivial.

Finally, let $\{W_j\}_{j \in J}$ be an open covering of M with $\overline{W}_j \subset V_j$. Then

$$\overline{W}_j \cap h_j\big(P_j \times (B_c \cap Q_j)\big)$$

is a closed subset of M with empty interior. As $j \in J$ and c vary we obtain a countable collection of closed sets with void interior. Their complement T in M is a dense G_δ. By the preceding paragraph, T consists of points with trivial holonomy.

Examples: To observe that T may have empty interior we construct the following example. Let $\alpha : T^2 \to T^2$ be the diffeomorphism given by the linear map $\tilde{\alpha} : R^2 \to R^2$ with $\tilde{\alpha} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The linear map $\tilde{\alpha}$ preserves two foliations of R^2, namely the straight lines parallel to one of the eigenvectors. Let F_1 and F_2 be the foliations of T^2 which result from projecting down to T^2 the two invariant foliations of R^2. Let M be the mapping torus of α. That is to say, M is a 3-dimensional manifold obtained from $T^2 \times R$ by identifying (x, t) with $(\alpha x, t+1)$ for each $x \in T^2$ and $t \in R$. Then M has a codimension-one foliation G which is a projection to M of the foliation $F_1 \times R$ of $T^2 \times R$. It is well known that the periodic points of α are a dense subset of T^2. A leaf of G passing through $(x, 0)$, where x is periodic, is easily seen to have non-trivial holonomy and the union of such leaves is dense.

We conclude by describing a modification of an example of Milnor [2] to give a codimension-one foliation of a non-paracompact, Hausdorff 3-dimensional manifold which has only one leaf, and that leaf has holonomy. The foliated manifold has charts $\{U_\alpha, h_\alpha\}_{\alpha \in R}$ with h_α a homeomorphism of R^3 onto U_α such that

$$h_\alpha(x_\alpha, y_\alpha, z_\alpha) = h_\beta(x_\beta, y_\beta, z_\beta),\ \alpha \neq \beta,$$

if and only if

(i) $x_\beta = x_\alpha \neq 0$ (this common value is denoted by x),

(ii) $y_\beta = y_\alpha + (\alpha - \beta)/x$,

(iii) $z_\beta = \begin{cases} z_\alpha + (\alpha - \beta), & x > 0 \\ 2^{\beta - \alpha} z_\alpha, & x < 0. \end{cases}$

We note that the associated changes of co-ordinates preserve the planes z = constant, $x > 0$ and z = constant, $x < 0$ so that M has a codimension-one foliation given by the condition z = constant. The associated holonomy maps are

(i) $x > 0 \quad g^+_{\beta\alpha}(z_\alpha) = z_\alpha + (\alpha - \beta)$,

(ii) $x < 0 \quad g^-_{\beta\alpha}(z_\alpha) = 2^{\beta - \alpha} z_\alpha$.

To see that there is only one leaf, note that

$$g^-_{0,\,z}\, g^+_{z,\,0}(\omega) = 2^{-z}(\omega - z).$$

Therefore in the chart with $\alpha = 0$, the leaf corresponding to $\omega = z$ is the same as the leaf corresponding to $z = 0$. This leaf has non-trivial holonomy since if $z \neq 0$, the above map has $z/(1 - 2^z)$ as a fixed point, and the derivative of $g^-_{0,\,z}\, g^+_{z,\,0}$ at this fixed point is 2^{-z}.

Remark. The easiest way to prove that the above manifold is Hausdorff is to use the continuous map to $\mathbb{R}^2$ which maps $(x_\alpha, y_\alpha, z_\alpha)$ to $(x_\alpha, x_\alpha y_\alpha + \alpha)$. This is consistent with the equivalence relation defining the manifold. Now use the fact that $\mathbb{R}^2$ is Hausdorff.

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