Comment. Math. Helvetici 61 (1986) 349–359 0010–2571/86/030349–11$01.50 + 0.20/0

Some evaluations of link polynomials

W. B. R. LICKORISH and K. C. MILLETT[1]

1. Introduction

For every oriented link L in the 3-sphere there is a 2-variable Laurent polynomial $P_L(\ell, m) \in \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$. It is defined uniquely by the formulae

(i) $P_U = 1$ for the unknot U;

(ii) $\ell P_{L_+} + \ell^{-1} P_{L_-} + m P_{L_0} = 0$, where L_+, L_-, and L_0 are any three links identical except within a ball where they are as shown in Figure 1. Details are given in [F–Y–H–L–M–O] and [L–M 1].

This two-variable polynomial is related to Δ_L, the Alexander polynomial, and V_L, the Jones polynomial, by

$$P_L(i, i(t^{1/2} - t^{-1/2})) = \Delta_L(t),$$
$$P_L(it^{-1}, -i(t^{1/2} - t^{-1/2})) = V_L(t).$$

The purpose of this paper is to evaluate P_L for various specific values of (ℓ, m), giving where possible the interpretation for V_L. The values chosen are such that P_L has an elementary form in terms of other known invariants of the link. Throughout, $c(L)$ denotes the number of components of L.

A few relevant elementary results that can be found in [J] or [L–M 1] are:

$$P_L(\ell, m) = P_L(-\ell, -m),$$
$$P_L(i, -2) = V_L(-1) = \Delta_L(-1) = \mathrm{Det}\,(L),$$
$$P_L(\ell, -(\ell + \ell^{-1})) = 1 = V_L(e^{-2\pi i/3}),$$
$$V_L(1) = (-2)^{c(L)-1}.$$

Let D_L and T_L denote the double and the treble cyclic covers of S^3, the 3-sphere, branched over L. Note that two of the expressions appearing above can

[1] Partially supported by U.S.A. National Science Grant DMS-8503733.

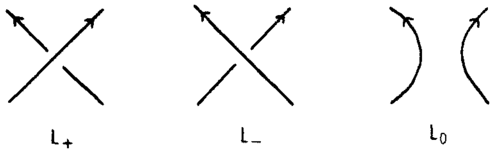

Fig. 1.

be expressed in terms of these covers namely

$$|\mathrm{Det}\,(L)| = \text{The order of } H_1(D_L; \mathbb{Z})$$

if Det $(L) \neq 0$ (in which case $H_1(D_L; Z)$ is infinite), and

$$c(L) - 1 = \text{Dimension } H_1(D_L; \mathbb{Z}_2).$$

The results that will be proved here are the following three theorems.

THEOREM 1 (H. Murakami [M])

$$P_L(1, \sqrt{2}) = V_L(i) = \begin{cases} (-\sqrt{2})^{c(L)-1}(-1)^{\mathrm{Arf}(L)} & \textit{if } \mathrm{Arf}\,(L) \textit{ exists,} \\ 0, & \textit{otherwise.} \end{cases}$$

THEOREM 2

$$P_L(1, 1) = (-2)^{1/2\ \textit{Dimension}\ H_1(T_L; \mathbb{Z}_2)}.$$

THEOREM 3

$$P_L(e^{i/\pi/6}, 1) = V_L(e^{i\pi/3}) = \pm i^{c(L)-1}(i\sqrt{3})^{\textit{Dimension}\ H_1(D_L; \mathbb{Z}_3)}.$$

The first theorem is included partly for completeness, but also because the short proof given here avoids knowledge of the connection between the Arf (or Robertello) invariant and the coefficients of the Conway potential function. It also produces, as a Corollary to Theorem 1, a very simple axiomatisation of the Arf invariant. Premonitions of Theorems 1 and 3 are to found in some of the results of V. F. R. Jones in [J] who, indeed, proved a version of Theorem 3 conjectured by J. S. Birman that did not identify the exponent of $\sqrt{3}$ appearing in the formula. Likewise A. Ocneanu conjectured that $P_L(1, 1)$ be a power of -2. During the preparation of this paper H. Murakami announced that he had proved Theorem 2.

It has long been known (see [W]) that there are inequalities relating the unknotting number of a knot and the dimensions of the homology groups of its cyclic branched covers. In the light of Theorems 2 and 3 it seems unlikely that new information about unkotting numbers (much sought from P_L) can be obtained from $P_L(1, 1)$ or $P_L(e^{i\pi/6}, 1)$, though calculation of these may give a quick way of computing two of the above mentioned dimensions. Similar remarks apply to considerations of bridge number and of braid index. It is amusing, for example, to note that for a rational, or two-bridge, link L, $P_L(1, 1)$ is always either 1 or -2.

2. $P_L(1, \sqrt{2})$

The Arf, or Robertello [R], invariant is defined on only the set $\mathcal{S}$ of oriented links for which each component has even linking number with the union of the other components.

Note. (a) If $L \in \mathcal{S}$, and $\hat{L}$ is constructed by banding together two distinct components of L, then $\hat{L} \in \mathcal{S}$.

(b) If $L \in \mathcal{S}$, and L' is formed by banding a component of L to itself and L'' is formed in exactly the same way only with one more complete twist in the band then precisely one of L' and L'' is in $\mathcal{S}$.

If α is a closed curve on a Seifert surface F of an oriented link L, let $q[\alpha]$ be the linking number modulo two of α and α-pushed-off-F. If $L \in \mathcal{S}$ (and not otherwise) this gives a well defined function

$$q : H_1(F; \mathbb{Z}_2)/i_*H_1(\partial F; \mathbb{Z}_2) \to \mathbb{Z}_2;$$

this q is a non-singular quadratic form.

DEFINITION. The Arf invariant of L, $\mathcal{A}(L)$, for $L \in \mathcal{S}$, is defined to be the value, 0 or 1, that q takes the more often.

PROPERTIES OF $\mathcal{A}$.

(i) *$\mathcal{A}$ is a well defined function $\mathcal{A} : \mathcal{S} \to \mathbb{Z}_2$.*
(ii) $\mathcal{A}(L_1 \# L_2) = \mathcal{A}(L_1) + \mathcal{A}(L_2)$.
(iii) $\mathcal{A}(\textit{Trefoil knot}) = 1$.
(iv) *If $L \in \mathcal{S}$, and $\hat{L}$ is constructed as in Note* (a), *then* $\mathcal{A}(L) = \mathcal{A}(\hat{L})$.

THEOREM 1 (H. Murakami). *Let L be an oriented link with $c(L)$ components.*

$$P_L(1, \sqrt{2}) = V_L(i) = \begin{cases} (-\sqrt{2})^{c(L)-1}(-1)^{\mathscr{A}(L)} & \text{if } L \in \mathscr{S} \\ 0 & \text{if } L \notin \mathscr{S}. \end{cases}$$

Proof. Let $A(L)$ denote $(-1)^{\mathscr{A}(L)}$ if $L \in \mathscr{S}$ and let $A(L)$ be zero otherwise. Suppose that L_+, L_-, and L_0 are oriented links identical except within a ball B where they are as in Figure 1.

CASE (i). Suppose that $c(L_+) < c(L_0)$. In this case both of L_+ and L_- belong to $\mathscr{S}$ or neither of them belongs to $\mathscr{S}$. If $L_0 \in \mathscr{S}$, Note (a) and Property (iv) imply that L_+ and L_- belong to $\mathscr{S}$ and all three have the same Arf invariant. Thus

$$A(L_+) + A(L_-) - 2A(L_0) = 0 \qquad (*).$$

This is trivially true when none of the three links is in $\mathscr{S}$. Thus there remains the possibility that $L_+ \in \mathscr{S}$, $L_- \in \mathscr{S}$, but $L_0 \notin \mathscr{S}$. However, the component of L_+ seen in Figure 1 can be banded to itself to produce X as in Figure 2(i). By Note (b) $X \in \mathscr{S}$, for adding a twist to that band would produce L_0. By Property (iv) $\mathscr{A}(X) = \mathscr{A}(L_+)$. As in Figure 2(ii), ($X$# (trefoil)) can have two of its components banded together to give L_-. Thus by Properties (ii), (iii) and (iv), $\mathscr{A}(L_+) + 1 = \mathscr{A}(L_-)$ modulo 2. Hence again (*) is satisfied.

CASE (ii). Suppose that $c(L_+) > c(L_0)$. If $L_0 \in \mathscr{S}$ then by Note (b) precisely one of L_+ and L_- is in $\mathscr{S}$ and that link has, by Property (iv), the same Arf invariant as L_0. If $L_0 \notin \mathscr{S}$ then by Note (a) neither L_+ nor L_- can be in $\mathscr{S}$. In either of these circumstances,

$$A(L_+) + A(L_-) - A(L_0) = 0 \qquad (**).$$

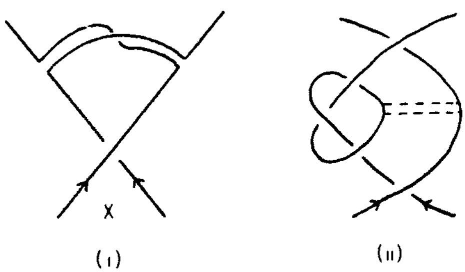

Fig. 2.

Now let $\hat{A}(L) = (-\sqrt{2})^{c(L)-1}A(L)$. The formulae (*) and (**) both become

$$\hat{A}(L_+) + \hat{A}(L_-) + \sqrt{2}\hat{A}(L_0) = 0.$$

Of course $\hat{A}(\text{unknot}) = 1$, so that $\hat{A}(L)$ and $P_L(1, \sqrt{2})$ satisfy the same defining relationships. Induction on the number of crossings of a link presentation shows at once that $\hat{A}(L) = P_L(1, \sqrt{2})$, and this completes the proof.

COROLLARY. *Properties* (i), (ii), (iii) *and* (iv) *of* $\mathscr{A}$ *given above can be taken as a complete set of axioms for the Arf, or Robertello, invariant of oriented links.*

Proof. Were there another invariant satisfying these properties it would, by the proof of Theorem 1, be related to $P_L(1, \sqrt{2})$ in exactly the same way as is the Arf invariant.

The result of Theorem 1 can be thought of as a resolution of the long standing mystery of why the Arf invariant is only defined on $\mathscr{S}$. Its generalisation to all oriented links can be thought of as the invariant $P_L(1, \sqrt{2})$.

3. $P_L(1, 1)$.

For oriented links L, $P_L(1, 1)$ is the integer defined in the usual way by

$$P_{L_+}(1, 1) + P_{L_-}(1, 1) + P_{L_0}(1, 1) = 0$$

and $P_U(1, 1) = 1$ where U denotes the unknot. It was conjectured by A. Ocneanu that $P_L(1, 1)$ be an integral power of -2. That is confirmed in what follows. For notation let d_L be the dimension as a vector space over $\mathbb{Z}_2$ of $H_1(T_L; \mathbb{Z}_2)$, where T_L is the three-fold cover of S^3 branched over L. The orientation of L means that T_L is well defined as the completion of the cover of $S^3 - L$ corresponding to the kernel of the map $\Pi_1(S^3 - L) \to Z_3$ that sends oriented meridians to 1. Then

THEOREM 2. *For any oriented link L in S^3,*

$$P_L(1, 1) = (-2)^{1/2d_L}.$$

In the proof of this theorem use will be made of the following two well known facts concerning arbitrary bounded 3-manifolds. Let M be a compact 3-manifold, and let $i:\partial M \to M$ be the inclusion of the boundary into M. Let K denote the kernel of $i_*:H_1(\partial M; Z_2) \to H_1(M; Z_2)$.

(a) $\operatorname{Dim} H_1(\partial M; Z_2) = 2 \dim K$.
(b) If $x \in K$ and $y \in K$ then $x \cdot y = 0$ where $x \cdot y$ is the modulo 2 intersection number of x and y.

The proof of (a) is a classical application of Poincaré-Lefschetz duality. For (b), regard x and y as 1-manifolds that bound mutually transverse surfaces in M; there must be an even number of end-points of the arcs of intersection of these surfaces.

Proof of Theorem 2. Let L_+, L_-, and L_0 be oriented links in S^3 identical outside a ball B in which they are as shown in Figure 3. The three diagrams that constitute Figure 3 are but variants of those of Figure 1; they are often more convenient when considering covers. Let M be the three-fold cyclic cover of $S^3 - B$ branched over $(S^3 - B) \cap L_j$. Then M is a 3-manifold, ∂M has genus 2 and, using the above notation, $\dim K = 2$. Further, Z_3 acts with generator ρ as the group of covering translations on M and K is invariant under ρ_*. Now $T_{L_j} = M \cup h_j$, where h_j is a handlebody of genus 2 being the three-fold cyclic cover of B branched over $B \cap L_j$. Consider a disc D properly embedded in B and separating the two components of $B \cap L_0$. Then D lifts to three discs in h_0 and the boundaries of these discs represent elements c_0, c_1, and c_2 of $H_1(\partial M; \mathbb{Z}_2)$, the notation being chosen so that $\rho_* c_k = c_{k+1}$ mod. 3. Note that $c_0 = c_1 + c_2$. The space of interest, $H_1(T_{L_0}; \mathbb{Z}_2)$ is the quotient of $H_1(M; \mathbb{Z}_2)$ by i_*C, where C is the space spanned by c_1 and c_2. Similarly $H_1(T_{L_+}; \mathbb{Z}_2)$ and $H_1(T_{L_-}; \mathbb{Z}_2)$ are quotients of $H_1(M; \mathbb{Z}_2)$ by i_*A and i_*B respectively, where A and B are the spaces spanned by $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Here $\{a_0, a_1, a_2\}$ and $\{b_0, b_1, b_2\}$ are elements of $H_1(\partial M; Z_2)$ represented by lifts of the boundaries of discs in B that separate the components of $B \cap L_+$ and $B \cap L_-$ respectively. The relative positions of curves representing these various classes on ∂M is shown in Figure 4, the notation being chosen so that $\rho_* a_k = a_{k+1}$ and $\rho_* b_k = b_{k+1}$ modulo 3. Note that $b_0 = a_0 + c_2$.

Because $\rho_* K = K$, either $K \cap A = \{0\}$ or $A \subset K$. Similarly, $K \cap B = \{0\}$ or $B \subset K$, and $K \cap C = \{0\}$ or $C \subset K$. Now, because

$$a_0 \cdot c_0 = b_0 \cdot c_0 = a_1 \cdot b_0 = 1,$$

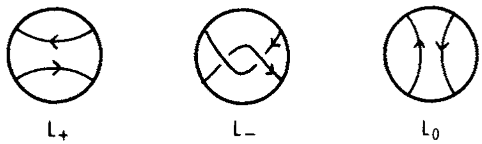

Fig. 3.

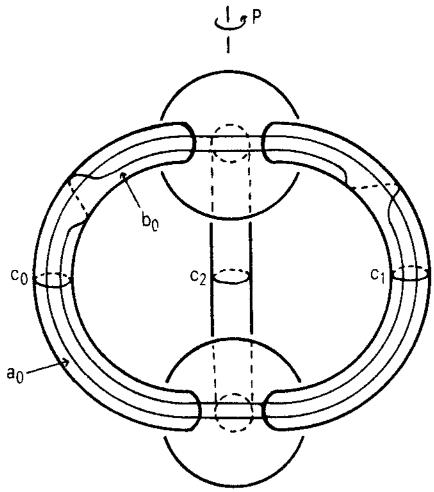

Fig. 4.

no *two* of the spaces A, B, and C can be contained in K (making use of (b)). Suppose that none of these spaces is in K: Then $K - \{0\}$ is in

$$H_1(\partial M; \mathbb{Z}_2) - (A \cup B \cup C) = \{a_0 + c_0, a_1 + c_1, a_2 + c_2\} \\ \cup \{a_0 + c_1, a_1 + c_2, a_2 + c_0\}$$

where the two triples on the right hand side of this expression are the two orbits under the Z_3 action. As K is invariant under the action, K must be the union of $\{0\}$ and one of these triples. However, by (b), this is not possible because

$$(a_0 + c_0) \cdot (a_1 + c_1) = 1 = (a_0 + c_1) \cdot (a_1 + c_2).$$

Thus of the spaces A, B, and C, precisely one is contained in K and each of the other two meet K in the zero element.

The numbers d_{L_+}, d_{L_-}, and d_{L_0} are the dimensions of the quotients of $H_1(M; \mathbb{Z}_2)$ by i_*A, i_*B, and i_*C respectively. Of course, K is the kernel of i_*, so, by the above analysis, one of these numbers is $\dim H_1(M; \mathbb{Z}_2)$ and the other two are two less than this. Hence

$$(-2)^{1/2 d_{L_+}} + (-2)^{1/2 d_{L_-}} + (-2)^{1/2 d_{L_0}} = 0.$$

Thus $(-2)^{1/2 d_L}$ satisfies the defining formula for $P_L(1, 1)$ and agrees with $P_L(1, 1)$ when L is the unknot. The usual induction on the number of crossings in a presentation for L finishes the proof.

4. $P_L(e^{i\pi/6}, 1)$.

The polynomial V_L of V. F. R. Jones is, for each oriented link L, related to P_L by the equation

$$V_L(t) = P_L(it^{-1}, -i(t^{1/2} - t^{-1/2}))$$

so that $V_L(e^{i\pi/3}) = P_L(e^{i\pi/6}, 1)$ and, in what follows, it will be preferable to work with the Jones polynomial. The reason for that is the reversing result for V_L:

The Jones reversing result. If $\hat{L}$ is obtained from L by reversing the orientation of one component that has linking number λ with the remaining components of L, then $V_{\hat{L}} = t^{-3\lambda}V_L$.

A proof of this can be found in [L–M 2] though beware that the conventions of that paper replace t by t^{-1}.

The reversing result leads to the "V_∞" formula first devised by J. S. Birman that will now be discussed. Here $c(L)$ denotes the number of components of a link L, and as usual L_+, L_-, L_0 are three oriented links identical except within a ball B where they are as in Figure 1.

PROPOSITION (J. S. Birman [B–K]). (i) *Suppose that* $c(L_+) < c(L_0)$. *Let* L_∞ *be obtained from* L_0 *by reversing one of the two components that meet* B (*with linking number* λ *with the rest of* L_0) *and banding it to the other as in Figure* 5(i). *Then*

$$t^{-1/2}V_{L_+} - t^{1/2}V_{L_-} + (t^{1/2} - t^{-1/2})t^{3\lambda}V_{L_\infty} = 0.$$

(ii) *Suppose that* $c(L_+) > c(L_0)$. *Let* L_∞ *be obtained from* L_+ *by reversing one of the components that meet* B (*which has linking number* μ *with the rest of* L_+) *and banding it to the other as in Figure* 5(ii). *Then*

$$t^{-1/2}V_{L_+} - t^{1/2}V_{L_-} + (t^{1/2} - t^{-1/2})t^{3(\mu-1/2)}V_{L_\infty} = 0.$$

Proof. Consider, as usual, a triple of links L_+, L_-, L_0 that are identical except within a ball B where they are as in Figure 1. The defining formula for the

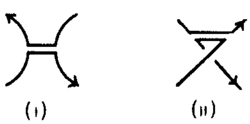

Fig. 5.

Jones polynomial is

$$t^{-1}V_{L_+} - tV_{L_-} + (t^{-1/2} - t^{1/2})V_{L_0} = 0, \qquad (1).$$

Case (i). Suppose that $c(L_+) < c(L_0)$.

Now consider the triple of links obtained by placing each of the tangles shown in Figure 6(a) inside B and using the same configuration as before in $S^3 - B$. Formula (1) applied to this new triple gives

$$t^{-1}V_X - tV_{L_0} + (t^{-1/2} - t^{1/2})V_{L_+} = 0, \qquad (2).$$

Reversing the direction of the components that meet B as the right-hand segments of the diagrams for L_0 and X leads to the situation of Figure 6(b). The reversing result implies that the Jones polynomials of $\hat{L}_0$ and $\hat{X}$ are $t^{-3\lambda}V_{L_0}$ and $t^{-3(\lambda+1)}V_X$. Thus Formula (1) applied to the triple of Figure 6(b) gives

$$t^{-1}V_{L_0} - t^{-2}V_X + (t^{-1/2} - t^{1/2})t^{3\lambda}V_{L_\infty} = 0, \qquad (3).$$

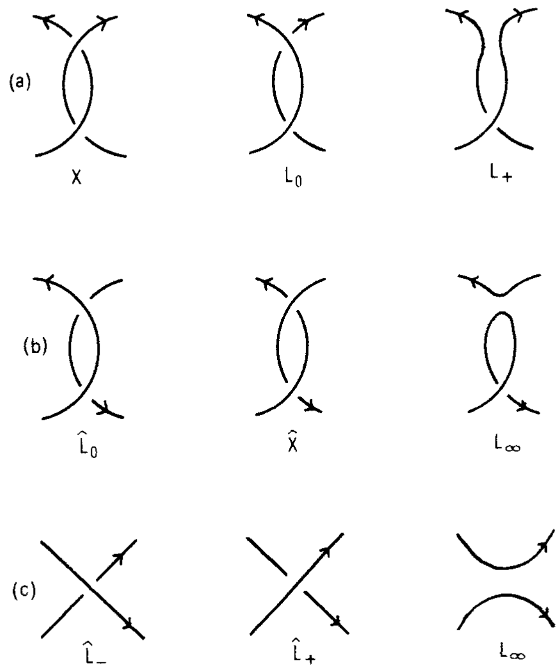

Fig. 6.

Then, the linear combination $t^{-1/2}(1)-t^{-1}(2)-(3)$ of the above formulae is the required result.

Case (ii) Suppose that $c(L_+)>c(L_0)$. Consider the links $\hat{L}_-$, $\hat{L}_+$, and L_∞ obtained by substituting the three tangles of Figure 6(c) into the ball B (this necessitates reversing one of the arcs in S^3-B). The Jones polynomials of $\hat{L}_-$ and $\hat{L}_+$ are $t^{-3(\mu-1)}V_{L_-}$ and $t^{-3\mu}V_{L_+}$ respectively. Applying Formula (1) to this triple of links gives

$$t^{-3(\mu-1)-1}V_{L_-}-t^{-3\mu+1}V_{L_+}+(t^{-1/2}-t^{1/2})V_{L_\infty}=0.$$

This is the required formula.

THEOREM 3. *Let L be an oriented link in S^3 with $c(L)$ components. Let D_L be the double cover of S^3 branched over L and let n_L be the dimension (quâ vector space) of $H_1(D_L;\mathbb{Z}_3)$. Then*

$$P_L(e^{i\pi/6},1)=V_L(e^{i\pi/3})=\pm i^{c(L)-1}(i\sqrt{3})^{n_L}.$$

[The general form of this result was conjectured by J. S. Birman and proved by V. F. R. Jones without identification of the integer n_L.]

Proof. Let L_+, L_-, and L_0 be a triple of oriented links as shown in Figure 1, and let L_∞ be that of Figure 5(i) if $c(L_+)<c(L_0)$ and that of Figure 5(ii) otherwise. Let $W_L=i^{(1-c(L))}V_L(e^{i\pi/3})$. Note that when $t=e^{i\pi/3}$, $(t^{1/2}-t^{-1/2})=i$ and $t^3=-1$. The latter implies, by way of the reversing result, that $(V_L(e^{i\pi/3}))^2$ is independant of the orientation of L. Now, with the sign ambiguity depending on whether or not $c(L_+)>c(L_0)$, the defining formula for V_L leads to

$$e^{-i\pi/3}W_{L_+}-e^{i\pi/3}W_{L_-}=\pm W_{L_0}.$$

The Proposition gives the following, where here the sign ambiguity depends on the parity of the linking numbers λ and μ:

$$e^{-i\pi/6}W_{L_+}-e^{i\pi/6}W_{L_-}=\pm iW_{L_\infty}.$$

Subtracting the square of this second equation from the square of the first (and using the fact that $e^{i2\pi/3}-e^{i\pi/3}=-1$) gives

$$(W_{L_+})^2+(W_{L_-})^2+(W_{L_0})^2+(W_{L_\infty})^2=0.$$

Now, in [B–L–M] a Laurent polynomial invariant $Q_L\in Z[x^{\pm1}]$ for unoriented links was *defined*, using the now familiar notation, by

$$Q_{L_+} + Q_{L_-} = x(Q_{L_0} + Q_{L_\infty})$$

and $Q_U = 1$ for the unknot U. Thus $(W_L)^2$ and $Q_L(-1)$ have identical defining formulae. The usual induction argument on the crossing number of a link presentation shows that $(W_L)^2 = Q_L(-1)$. However it is proved in [B–L–M], Property 5, that $Q_L(-1) = (-3)^{n_L}$ and this completes the proof of the theorem.

Remark. The proof in [B–L–M] uses no special theory of the Q_L polynomial to show that $Q_L(-1) = (-3)^{n_L}$. It is simply shown that n_L is the nullity of a certain symmetric matrix over Z_3 associated with a (generalised) Seifert form for L. The nullities for L_+, L_-, L_0, and L_∞ are easily shown to be of the form n, n, n, and $(n+1)$ in *some* order. So, it is immediate that $(-3)^{n_L}$ satisfies the defining formulae for $Q_L(-1)$.

REFERENCES

[B–K] J. S. BIRMAN and T. KANENOBU. *Jones' braid-plat formulae, and a new surgery triple.* (to appear).

[B–L–M] R. D. BRANDT, W. B. R. LICKORISH and K. C. MILLETT. *A polynomial invariant for unoriented knots and links,* Invent. Math. (to appear).

[F–Y–H–L–M–O] P. FREYD, D. YETTER, J. HOSTE, W. B. R. LICKORISH, K. MILLETT and A. OCNEANU. *A new polynomial invariant of knots and links,* Bull. A.M.S., *12* (1985), 239–246.

[J] V. F. R. JONES. *A polynomial invariant for knots via von Neumann algebras,* Bull. A.M.S., *12* (1985), 103–111.

[L–M 1] W. B. R. LICKORISH and K. C. MILLETT. *A polynomial invariant of oriented links,* Topology (to appear).

[L–M 2] W. B. R. LICKORISH and K. C. MILLETT. *The reversing result for the Jones polynomial,* Pacific J. Math. (to appear).

[M] H. MURAKAMI. *A recursive calculation of the Arf invariant of a link,* (to appear).

[R] R. A. ROBERTELLO. *An invariant of knot cobordism,* Comm. Pure Appl. Math., *18* (1965), 543–555.

[W] H. WENDT. *Die gordische Auflösung von Knoten,* Math. Zeit. *42* (1937) 680–696.

Department of Pure Mathematics,
16, Mill Lane,
Cambridge, CB2 1SB,
England.

Department of Mathematics,
University of California,
Santa Barbara,
CA 93106, U.S.A.

Received September 30, 1985