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A polynomial invariant for unoriented knots and links

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§ 1. Introduction

The recent discovery by V.F.R. Jones [J] of a link polynomial, which complements the classical Alexander polynomial, has been generalized by the discovery of a new two-variable Laurent polynomial $P(L)$ associated with an oriented classical link L (see [L-M] and [F-Y-H-L-M-O]). This polynomial is calculated by a recursive formula involving changing crossings in a presentation of the link until the unlink is obtained. Here yet another (Laurent one-variable) polynomial $Q(L)$ will be associated to an *unoriented* link L in a manner that is spiritually very close to the way of defining the above mentioned two-variable polynomial. These polynomials are, however, different in the sense that there are pairs of knots distinguished by one polynomial but not the other. The theorem to be proved in this paper is as follows:

Theorem. *There is a unique function Q from the set of unoriented links of S^1's in S^3 to $\mathbf{Z}[x^{\pm 1}]$ such that:*

(i) *$Q(L)$ depends only on the isotopy class of L;*

(ii) *If U is the unknot, then $Q(U)=1$;*

(iii) *If L_+, L_-, L_0, and L_∞ are links that are identical except near one point where they are as in Fig.* 1,

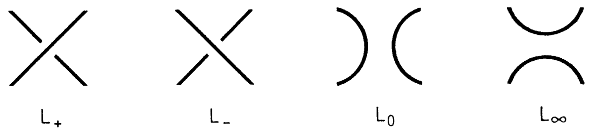

Fig. 1

then

$$Q(L_+)+Q(L_-)=x(Q(L_0)+Q(L_\infty)). \qquad (*)$$

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The lack of orientation makes it unclear which diagram in Fig. 1 should be labeled L_+ and which L_-, with similar confusion existing between L_0 and L_∞. However the symmetry of the formula neutralises this ambiguity. As in [L-M] it is easy to show that if such a function exists it is unique, so the proof of the theorem, which constitutes the next section, concentrates entirely on existence. Similarly it is easy to deduce that, if U^c denotes the unlink of c components, then $Q(U^c)=\mu^{c-1}$ where $\mu=2x^{-1}-1$.

Some properties of Q will be proved in §3. In particular, for any link L, $Q(L)(1)=1$, $Q(L)(-1)=(-3)^d$, $Q(L)(-2)=(-2)^{c-1}$, and $Q(L)(2)=(\delta_L)^2$, where c is the number of components of L, d is the dimension of the mod 3 homology of the double cover of S^3 branched along L, and δ_L is the determinant of L. These last results do at least correlate $Q(L)$ with other known invariants of L.

The basic structure of linear skein theory carries over to $Q(L)$ with some obvious modifications. Many of the examples in [L-M] involved 2×2 matrices whereas their replacement in the new theory requires 3×3 matrices, the results on the numerators and denominators and on rational knots are thus more complicated. It is intended to give a brief survey of these formulae and a more comprehensive table of polynomials in another paper. The following results are, however, elementary consequences of the theorem or of linear skein theory.

Property 1. (a) *$Q(L_1 \# L_2)=Q(L_1)Q(L_2)$ where $L_1 \# L_2$ denotes any 'connected' sum of L_1 and L_2.*

(b) *$Q(L_1\cup L_2)=\mu Q(L_1)Q(L_2)$ where $L_1\cup L_2$ is the 'distant' union of L_1 and L_2.*

(c) *$Q(L)=Q(\bar{L})$ where $\bar{L}$ is the mirror image of L.*

(d) *If L_2 is a mutant of L_1* (see [L-M]), *then $Q(L_1)=Q(L_2)$.*

This new polynomial *seems* to be the only new polynomial that can be defined by a perturbation of the ideas and methods of [L-M]. Other ideas for a new polynomial have usually turned out to give the original two-variable polynomial modified in some way by knowledge of the number of components of the link. The fact that 'L_∞' occurs in the definition of Q does suggest that Q might be independent of the two-variable polynomial, and an example will be given to show that that is indeed so.

An announcement has been made by C.F. Ho of his independent discovery of the new polynomial [H].

§ 2. The proof of the theorem

The proof of the Theorem will entail an inductive definition of Q and a sequence of lemmas; it will be similar to that given in [L-M] so emphasis will be given to points at which the proofs differ.

Let $\mathscr{L}_n$ denote the set of all (generic) projections, with at most n crossings, of links equipped with an ordering $(c_1, c_2, c_3, \ldots)$ of their components and with a basepoint b_i and an orientation for each component c_i. This information induces an ordering on all the points of $L\in\mathscr{L}_n$; the ordering begins at the basepoint b_1 of c_1, proceeds along all of c_1 in the given direction and then transfers to the basepoint b_2 of c_2, and so on. If $L\in\mathscr{L}_n$, the *standard ascending projection* $\alpha L\in\mathscr{L}_n$ associated to L is the projection formed by switching the crossings of L (from over to under or vice versa) so that, proceeding along L in the above ordering of points, each crossing is first encountered as an undercrossing.

If a link is regarded as a subset of $\mathbf{R}^2\times\mathbf{R}$, the projection $\mathbf{R}^2\times\mathbf{R}\to\mathbf{R}$ defines a height function on the link. Here links are being regarded as subsets of the plane that are images of immersed 1-manifolds with over-crossing information recorded. Height functions will now be used on such link projections, these being two-valued functions at the cross-over points with the obvious definition of continuity.

Definition. Let $L\in\mathscr{L}_n$. A continuous function $h\colon L\to\mathbf{R}$ is an *untying function* if:

(i) $y_i\in c_i$, $y_j\in c_j$ and $i<j$ implies that $h(y_i)<h(y_j)$;

(ii) On each c_i the function h is monotonically increasing from the basepoint b_i to some (top)point t_i, and is monotonically decreasing from t_i to b_i;

(iii) At a cross-over point the value of h at the over-pass exceeds that at the under-pass.

Note. (a) Any standard ascending projection has an untying function.

(b) If L has an untying function, it represents an unlink.

Inductive Hypothesis $(n-1)$. Suppose that a function

$$Q\colon \mathscr{L}_{n-1}\to\mathbf{Z}[x^{\pm 1}]$$

has been defined such that:

(i) Q is independent of choice of basepoints, of ordering of components and of orientations;

(ii) Q is invariant under Reidemeister moves that do not increase the number of crossings beyond $(n-1)$;

(iii) If $L_+\in\mathscr{L}_{n-1}$, then with the usual notation,

$$Q(L_+)+Q(L_-)=x(Q(L_0)+Q(L_\infty));$$

(iv) If $L\in\mathscr{L}_{n-1}$ and L has an untying function, then $Q(L)=\mu^{c-1}$, where c is the number of components of L and $\mu=2x^{-1}-1$.

Of course any element of $\mathscr{L}_0$ has an untying function, so that Q is *defined* on $\mathscr{L}_0$ by (iv), and this starts the induction.

Recursive Definition (n). If $L\in\mathscr{L}_n$ define $Q(\alpha L)$ to be μ^{c-1} where L has c components. If L and αL differ at at most $(r-1)$ crossings assume that $Q(L)$ has been defined. If now they differ at r crossings let $Q(L)$ be the polynomial calculated by applying the formula $(*)$,

$$Q(L_+)+Q(L_-)=x(Q(L_0)+Q(L_\infty))$$

to the first such crossing encountered in proceeding along L from the basepoint b_1. Note that although here L_0 and L_∞ are not well defined as elements of $\mathscr{L}_{n-1}$ since order, orientations, and basepoints are missing, $Q(L_0)$ and $Q(L_\infty)$ are well defined (by induction on n). By convention, L_0 will be chosen to denote the link obtained by nullifying the relevant crossing of L_+ (or L_-) in the manner *consistent* with orientations.

This now defines Q on $\mathscr{L}_n$, and it will now be checked, in a sequence of lemmas, that Q satisfies the induction hypothesis (n).

Lemma 1(*n*). *Suppose $L \in \mathscr{L}_n$. Suppose that the crossings where L and αL differ are labelled in any sequence. The polynomial corresponding to L, calculated from $Q(\alpha L)$ by applying formula $(*)$ to the crossings in the sequence, is $Q(L)$.*

Proof. By induction on the number of crossing differences between L and αL, it is only necessary to consider the effect of exchanging the order of crossings labelled "i" and "j". Let $\sigma_i L$, $\eta_i^0 L$, and $\eta_i^\infty L$ be "L" with the crossing labelled "i" switched, and nullified in the two possible ways. Considering i followed by j, the polynomial calculated for L is $-Q(\sigma_i L)+x\{Q(\eta_i^0 L)+Q(\eta_i^\infty L)\}$, which, by changing crossing j, becomes equal to

$$\begin{aligned} &Q(\sigma_j\sigma_i L)-x[Q(\eta_j^0\sigma_i L)+Q(\eta_j^\infty\sigma_i L)]+x\{-Q(\sigma_j\eta_i^0 L)+x[Q(\eta_j^0\eta_i^0 L) \\ &\quad +Q(\eta_j^\infty\eta_i^0 L)]-Q(\sigma_j\eta_i^\infty L)+x[Q(\eta_j^0\eta_i^\infty L)+Q(\eta_j^\infty\eta_i^\infty L)]\}. \end{aligned}$$

As the σ, η^0, and the η^∞ operations all commute this formula is, by inspection, symmetric in i and j. Hence considering the two crossings in the other order gives the same polynomial for L. □

Lemma 2(*n*). *$Q|\mathscr{L}_n$ is independent of choice of basepoints.*

Proof. We need only show that if a basepoint of a component lies on a segment of the projection it can be moved to an adjacent segment without changing the polynomial. Suppose the basepoint on component c_i is to be changed from position b_1 to position b_2, passing a crossing of c_i with c_j. Let L_1 and L_2 denote the relevant elements of $\mathscr{L}_n$ that have basepoints on c_i at b_1 and b_2, respectively, and are otherwise exactly the same.

Case $i \neq j$. Here αL_1 and αL_2 represent exactly the same projection though with different basepoints. Thus $Q(L_1)=Q(L_2)$ as, by Lemma 1(n), the choice of the order in which the sequence of crossing changes is accomplished does not change the polynomial.

Case $i=j$. In this case αL_1 and αL_2 differ only at the crossing under consideration where the associated crossing changes are labelled σ, η^0, and η^∞. By Lemma 1(n), $Q(L_1)$ can be calculated by first changing all other relevant crossings and thereby giving $Q(L_1)=f(Q(\sigma\alpha L_1))$, where f is some linear function. Similarly, $Q(L_2)=f(Q(\alpha L_2))$, with exactly the same function f occuring because the calculation involves only projections of fewer crossings for which (by induction) the position of the basepoints is irrelevant. By definition, $Q(\alpha L_1)$ and $Q(\alpha L_2)$ are equal to μ^{c-1} where c is the number of components. Furthermore $\eta^0\alpha L_1 \in \mathscr{L}_{n-1}$ is ascending and has $c+1$ components and therefore (by induction) has polynomial μ^c.

Now $\eta^\infty \alpha L_1$, the projection obtained from αL_1 by nullifying the crossing with a miss-match of orientations, has $(n-1)$ crossings and c components. But, removal of a neighborhood of the crossing from c_i splits c_i into two parts, and the untying function associated (by ascendingness) to αL_1 restricts to monotone functions on these two parts. These monotone functions combine to give an untying function for $\eta^\infty \alpha L_1$. Thus the induction implies that $Q(\eta^\infty \alpha L_1) = \mu^{c-1}$. Therefore

$$\begin{aligned} Q(\sigma\alpha L_1) &= -Q(\alpha L_1) + x(Q(\eta^0 \alpha L_1) + Q(\eta^\infty \alpha L_1)) \\ &= -\mu^{c-1} + x(\mu^c + \mu^{c-1}) = \mu^{c-1}. \end{aligned}$$

Direct substitution in "f" shows that $Q(L_1) = Q(L_2)$. □

Lemma 3(n). *$Q|\mathscr{L}_n$ satisfies formula* (*).

Proof. Suppose, with the usual notation, L_+, L_-, L_0, and L_∞ are in $\mathscr{L}_n$. The formula

$$Q(L_+) + Q(L_-) = x(Q(L_0) + Q(L_\infty))$$

is the first step in a calculation (permitted by Lemma 1(n)) of either $Q(L_+)$ from $Q(\alpha L_+)$ or of $Q(L_-)$ from $Q(\alpha L_-)$, depending upon which of $L_\pm$ differs from $\alpha L_\pm$ at the crossing under consideration. □

Lemma 4(n). *Suppose that L is an element of $\mathscr{L}_n$ with c components that has an untying function h: $L \to \mathbf{R}$. Then $Q(L) = \mu^{c-1}$.*

Proof. The link projection L has components $\{c_i\}$ each with an orientation, a basepoint b_i and a top-point t_i. Suppose $v = \sum_1^c v_i$, where v_i is the number of self-crossings of c_i on the segment from t_i to b_i. If $v = 0$ the result is true as L is then a standard ascending projection. Thus, inductively, assume the result true for any $L \in \mathscr{L}_n$ with an untying function and a lower value of v than the one given.

Let c_i be a component of L for which $v_i > 0$. Consider the first self-crossing X of c_i after t_i (in the direction specified by the orientation). If that crossing is an over-pass (as encountered proceeding from t_i) the function h may be changed to be increasing on the segment from t_i to X and just beyond X. This produces a smaller value of v, so that $Q(L) = \mu^{c-1}$ by induction on v.

Thus assume the segment from t_i encounters X as an under-pass. The fact that h decreases from t_i to b_i implies that this will be an underpassing of the segment from b_i to t_i. Let σL, $\eta^0 L$, and $\eta^\infty L$ be L with crossing X switched or nullified in each of the two possible ways. The situation is depicted in Fig. 2(a) where L is shown with an unbroken line where h is increasing and a broken line where h is decreasing. In Fig. 2(b) σL is shown and, reasoning as in the preceding paragraph shows that $Q(\sigma L) = \mu^{c-1}$. As shown in Fig. 2(c), $\eta^0 L$ has components c_i' and c_i'' (containing b_i and t_i, respectively) in place of c_i, and c_i'' always crosses c_i' as an over-pass. The function h can be adjusted, as indicated by the broken lines, to be an untying function for $\eta^0 L$ so that, by induction on n, $Q(\eta^0 L) = \mu^c$. Finally, h can be adjusted, as shown in Fig. 2(d), to be an

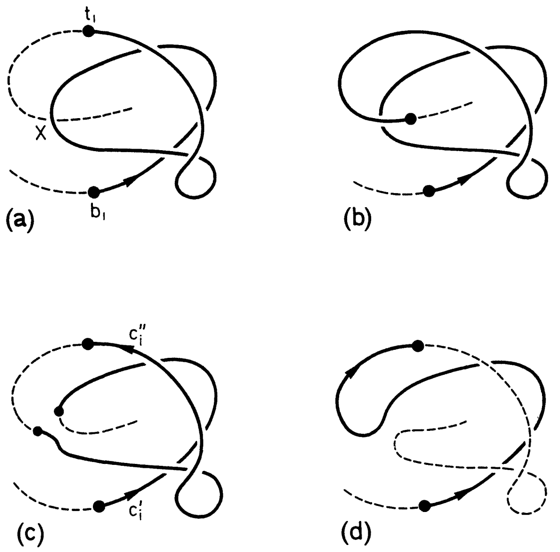

Fig. 2

untying function of $\eta^{\infty} L$ so that $Q(\eta^{\infty} L)=\mu^{c-1}$. Thus, using Lemma 3($n$),

$$\begin{aligned} Q(L) &= -Q(\sigma L)+x[Q(\eta^0 L)+Q(\eta^{\infty} L)] \\ &= -\mu^{c-1}+x[\mu^c+\mu^{c-1}]=\mu^{c-1}. \end{aligned}$$

This completes the induction on v, and hence establishes the lemma. □

Corollary 4.1(n). *$Q|\mathscr{L}_n$ is independent of choice of orientations of components.*

Proof. Let $L\in\mathscr{L}_n$ and let L' be L with the orientation of one component c_i reversed. Let βL be $\alpha L'$ with the i^{th} orientation changed back again. But βL clearly has an untying function, so $Q(\beta L)\equiv\mu^{c-1}$.

The calculation of $Q(L')$ from $Q(\alpha L')\equiv\mu^{c-1}$ is the same as that for $Q(L)$ from $Q(\beta L)\equiv\mu^{c-1}$, so that $Q(L')=Q(L)$. □

Lemma 5(n). *$Q(L)$ is invariant under Reidemeister moves that do not increase the number of crossings beyond n.*

Proof. The proof of Proposition 4(n) of [L-M] translates immediately to give a proof of this lemma. □

Proof of the Theorem. The theorem is proved once the induction hypothesis (n) is established. It only remains to prove that $Q(L)$ is independent of choice of ordering for $L\in\mathscr{L}_n$. This is not obvious. The proof is exactly Propositions 5(n) and 6(n) of [L-M] with the symbols "Q" and "L" replacing "P" and "K" throughout. □

Remark (J.H. Conway). Propositions 5(n) and 6(n) of [L-M] can be thought of as proving that any ascending element of $\mathscr{L}_n$ can be changed to an element of $\mathscr{L}_0$ by a sequence of basepoint changes and Reidemeister moves that do not increase the number of crossings beyond n, provided the move of Fig. 3 is included.

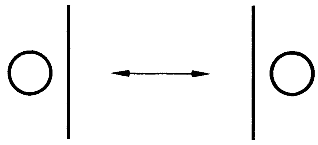

Fig. 3

§3. Basic properties of the polynomials

The statement of Property 1 in the introduction recorded the behaviour of the new polynomial under the link operations of connected sum, distant union, reflection and mutation. It follows that the polynomial of the c-component unlink is μ^{c-1} where $\mu=2x^{-1}-1$. Further properties of an elementary nature are listed below.

Property 2. *For any link L, $Q(L)-1$ is divisible by $2(x-1)$.*

Proof. Using induction, on the number of crossings in a presentation and the number of crossings that must be changed to achieve the ascending configuration, this follows from

$$Q(L_+)-1+Q(L_-)-1=x\{Q(L_0)-1+Q(L_\infty)-1\}+2(x-1). \quad \square$$

Corollary. (i) $QL(1)=1$.

(ii) *In $Q(L)$ the constant term is odd, and all the other coefficients are even.*

Property 3. *If L has c components, $QL(-2)=(-2)^{c-1}$.*

Proof. The method of proof is a similar double induction using the formula (*). $\square$

Property 4. *For any link L, $QL(2)=(\delta_L)^2$ where δ_L is the determinant of L.*

A little discussion is in order before proving this result. The determinant of L, δ_L, is a classical invariant of L which may be defined to be the modulus of the Alexander polynomial of L evaluated at -1. Then δ_L is the order of the first homology group of the double cover of S^3 branched over L (that group being infinite when δ_L is zero). This is attractively independent of any choice of orientation.

Because the theory of Q is allergic to orientation it is desirable to work with a definition of δ_L that is comparitively free of orientations. Such an approach was classically given by the fact that δ_L is the modulus of the determinant of the Goeritz matrix associated to any presentation of L, [G]. A new interpretation of the Goeritz method was given by C.McA. Gordon and R.A. Litherland [G-L]. To any (maybe unorientable) connected surface V spanning L they associate a symmetric bilinear form $\mathcal{G}_V$: $H_1(V)\times H_1(V)\to\mathbf{Z}$. The absolute value of the determinant of this bilinear form is an invariant of the link; it is δ_L, for, by judicious choices of V and a base of $H_1(V)$, $\mathcal{G}_V$ is represented by a Goeritz matrix. Suppose that α, $\beta\in H_1(V)$ are represented by 1-cycles a, b. Then $\mathcal{G}_V(\alpha,\beta)$ is defined to be the linking number of a and τb, where τb is $2b$ pushed off V into $S^3 - V$, the push-off being locally to both sides of V. This characterisation of δ_L will now be used.

Proof of Property 4. First note that $QU(2) = 1 = (\delta_U)^2$ where U is the unknot. Now $QL(2)$ can be calculated with no ambiguity in the usual recursive way (inducting on the number of crossings) from $QL_+(2) + QL_-(2) = 2(QL_0(2) + QL_\infty(2))$, where $QU(2) = 1$ for U the unknot. Here $L_\pm$, L_0 and L_∞ are links with projections related in the usual way. Thus the property follows at once if it can be established that $(\delta_{L_+})^2 + (\delta_{L_-})^2 = 2((\delta_{L_0})^2 + (\delta_{L_\infty})^2)$. Figure 4 shows the links L_i for $i = +, -, 0, \infty$, each with a connected spanning surface V_i shaded in.

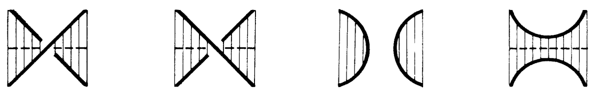

Fig. 4

These surfaces can be constructed by using Seifert's circuit method for V_0, connecting up components with hollow handles away from the area depicted, and modifying as shown to obtain V_+, V_-, and V_∞. Thus all but V_∞ could be taken to be orientable. The four surfaces are identical outside the area shown. Take a base for $H_1(V_0)$ represented by oriented (simple) closed curves on V_0 and for bases of $H_1(V_i)$, $i = +, -, \infty$, take the homology classes of the dotted curves in the diagrams in Fig. 4 (the ends of which join up outside the diagram) and the set of curves already chosen on V_0. The matrices M_i that represent the $\mathcal{G}_{V_i}$ with respect to these bases have the form

$$M_\infty = \begin{bmatrix} n & \rho \\ \rho^\tau & M_0 \end{bmatrix}, \quad M_\pm = \begin{bmatrix} n \mp 1 & \rho \\ \rho^\tau & M_0 \end{bmatrix}.$$

Thus $\det M_\pm = \det M_\infty \mp \det M_0$, and so squaring and adding one obtains

$$(\det M_+)^2 + (\det M_-)^2 = 2((\det M_\infty)^2 + (\det M_0)^2).$$

Recollection that $\delta_{L_i} = |\det M_i|$ yields the required formula. □

The next result was conjectured by J.H. Conway.

Property 5. *$QL(-1) = (-3)^d$, where $d = \dim H_1(D; \mathbf{Z}_3)$ and D is the double cover of S^3 branched over L.*

Proof. In the notation of the previous proof, let D_i be the double cover of S^3 branched over L_i and let $d_i = \dim H_1(D_i; \mathbf{Z}_3)$. $H_1(D_i; \mathbf{Z}_3)$ is presented by $\hat{M}_i$, the matrix M_i reduced modulo 3, so that d_i is the nullity of $\hat{M}_i$. The form of the matrices $\hat{M}_i$ for $i = +, -, 0$, and ∞ shows that three of their nullities take a common value a, the fourth is $a+1$. Hence $(-3)^{d_+} + (-3)^{d_-} + (-3)^{d_0} + (-3)^{d_\infty} = 0$, and the usual induction argument completes the proof. □

Remark. The determinant δ_L can be calculated, up to sign, as $\det(A + A^\tau)$ where A is a Seifert matrix for L. The Alexander polynomial is $\det(t^{1/2} A - t^{-1/2} A^\tau)$. It is natural to wonder whether, in the light of the above, $Q(L)$ could be defined from some slight generalisation of $\mathscr{G}_V$.

Property 6. *If L has c components the lowest power of x in $Q(L)$ is precisely $(1-c)$.*

Proof. The usual double induction procedure shows that $(1-c)$ is a lower bound for the powers of x in $Q(L)$. Thus, by the Corollary to Property 2, this result is true when $c=1$. Now induction, on c and on the number of crossing changes necessary to change L to the distant union of its components, produces the result in general. □

Property 7. *If L has c components,*

$$[x^{c-1}Q(L)]_{x=0}=[(-m)^{c-1}P(L)]_{(l,m)=(1,0)}$$

where $P(L)$ is the two-variable polynomial in l and m defined in [L-M].

Proof. The result is trivial if L is the unlink. The substitution $(l,m)=(1,-y)$ changes $P(L)$ to a Laurent polynomial in y defined in the usual way by $P(L_+)+P(L_-)=yP(L_0)$ and $P(U)=1$. Now $x^{c-1}Q(L)$ and $y^{c-1}P(L)$ are genuine polynomials with no negative powers of x and y. Consider the defining formulae for Q and P when $L_\pm$ has c components:

Case (i). L_0 has $c+1$ components and L_∞ has c components.

$$x^{c-1}QL_+ + x^{c-1}QL_- = x^cQL_0 + x(x^{c-1}QL_\infty)$$

$$y^{c-1}PL_+ + y^{c-1}PL_- = y^cPL_0.$$

Case (ii). L_0 and L_∞ each have $(c-1)$ components.

$$x^{c-1}QL_+ + x^{c-1}QL_- = x^2(x^{c-2}QL_0 + x^{c-2}QL_\infty)$$

$$y^{c-1}PL_+ + y^{c-1}PL_- = y^2(y^{c-2}PL_0).$$

The result now follows, substituting $x=0=y$, from the usual induction on the number of crossings and the number of crossing differences from the standard ascending projection. □

Property 8. *The degree of $Q(L)$ is less than the crossing number of L.*

Proof. This follows from the usual double induction using formula $(*)$. □

Examples. Employing the standard notation for knots and links having presentations with few crossings;

$$Q(8_8)=1+4x+6x^2-10x^3-14x^4+4x^5+8x^6+2x^7$$

$$Q(10_{129}\text{ reflected})=1-12x-2x^2+26x^3+4x^4-20x^5-4x^6+6x^7+2x^8$$

$$Q(13_{6714})=1+20x+14x^2-62x^3-40x^4+64x^5+38x^6$$
$$-26x^7-14x^8+4x^9+2x^{10}.$$

Calculations for such examples are best performed by computer; in these examples, however, human verification was also employed. The interest of the above example is that the three given knots have the same two-variable polynomial (see [L-M] Example 16). Thus the Q polynomial is *not* a function of the two-variable polynomial (and so certainly not just a function of the polynomials of Alexander and Jones).

Table. Values of the Q-polynomial

3_1	$-3+2+2$
4_1	$-3-2+4+2$
5_1	$5-2-6+2+2$
5_2	$1-4-2+4+2$
6_1	$1+4-6-4+4+2$
6_2	$5-2-10+0+6+2$
6_3	$5-6-12+4+8+2$
7_1	$-7+4+16-6-10+2+2$
7_2	$-3+6+8-10-6+4+2$
7_3	$-3+2+6-6-4+4+2$
7_4	$1+8-4-12+0+6+2$
7_5	$1+0-4-6+2+6+2$
7_6	$5+2-12-10+6+8+2$
7_7	$5+6-18-14+10+10+2$
8_1	$-3-6+14+12-14-8+4+2$
8_2	$-7+0+22+2-20-4+6+2$
8_3	$1-8+4+12-8-6+4+2$
8_4	$-3+2+14-2-16-2+6+2$
8_5	$-11+14+26-16-24+2+8+2$
8_6	$1-4+2+2-8+0+6+2$
8_7	$-7+4+20-8-20+2+8+2$
8_8	$1+4+6-10-14+4+8+2$
8_9	$-7+4+16-10-16+4+8+2$
8_{10}	$-11+14+22-22-22+8+10+2$
8_{11}	$-3+6+4-12-10+6+8+2$
8_{12}	$5+2-8-12-4+8+8+2$
8_{13}	$-3+10+10-22-16+10+10+2$
8_{14}	$1+8+0-22-10+12+10+2$
8_{15}	$-7+16+10-32-16+16+12+2$
8_{16}	$-3+10+18-22-30+8+16+4$
8_{17}	$-3+6+12-20-24+10+16+4$
8_{18}	$5+2+12-26-36+14+24+6$
8_{19}	$-11+10+20-10-12+2+2$
8_{20}	$-7+12+12-14-8+4+2$
8_{21}	$-7+8+6-12-2+6+2$
Conway-Kinoshita-Terasaka	$17-24-52+54+76-28-48-4+8+2$
$(-3,5,7)$-pretzel	$1+48-72-172+234+256-286-206+162+94-42-22+4+2$
untwisted double of trefoil	$17+8-180+134+556-618-872+978+818-736-470+284$ $+158-54-28+4+2$
2_1^2	$-2x^{-1}+1+2$
4_1^2	$2x^{-1}-1-4+2+2$
5_1^2	$2x^{-1}-1-8+0+6+2$
6_2^3	$4x^{-2}-4+1+0-16+0+12+4$

To give a feel for the nature of this new polynomial, a small table is given above in which the Laurent polynomial $\sum_{-r}^{s} a_i x^i$ is written $a_{-r}x^{-r}+a_{-r+1}+a_{-r+2}+\ldots+a_s$, with $a_0 x^0$ written a_0. The usual name of a knot or link preceeds the polynomial.

Direct calculation has also shown that the two knots of [L-M] Example 17, which have distinct signatures (and so are not mutants) have the same Q-polynomial, i.e. $-7+4x+16x^2-6x^3-10x^4+2x^5+2x^6$.

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