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A POLYNOMIAL INVARIANT OF ORIENTED LINKS

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§0. INTRODUCTION

THE THEORY of classical knots and links of simple closed curves in the 3-dimensional sphere has, for very many years, occupied a pre-eminent position in the theory of low dimensional manifolds. It has been a motivation, an inspiration and a basis for copious examples. Knots have, in theory, been classified by Haken [10] but the classification is by means of an algorithm that is too complex to use in practice. Thus one is led to seek simple invariants for knots which will distinguish large classes of specific examples. A knot (or link) invariant is a function from the isotopy classes of knots to some algebraic structure. Perhaps the most famous invariant of a knot K is the Alexander polynomial, $\Delta_K(t)$, a Laurent polynomial in the variable t. This was introduced by Alexander [1] who explained how to calculate the polynomial by taking the determinant of a matrix associated with a presentation (or picture) of the knot given by a suitably chosen projection of its spatial position to a plane. The Alexander polynomial is still remarkably efficacious in distinguishing specific knots and, being readily calculable by computer, is employed by modern compilers of prime knot tables as the fundamental invariant to distinguish between examples (see Thistlethwaite [20]). Of course other invariants, notably signatures and the sophisticated Casson–Gordon 'invariants' are now available as well. Nevertheless, $\Delta_K(t)$ is still a most useful invariant. Much has been written on this polynomial during the last sixty years; and a modern definition might be as follows. If X is $S^3 - K$, where K is now an oriented link, let X_∞ be the covering of X corresponding to the kernel of the homomorphism $\pi_1(X) \to H_1(X) \to \mathbb{Z}$ that sends meridians (with prefered orientation) to 1. Thus X_∞ is acted upon by the infinite cyclic group, which is to be considered as a multiplicative group with generator denoted by t, acting as the deck transformations of the covering space. Then $H_1(X_\infty)$ is a finitely generated module over the ring $\mathbb{Z}[t, t^{-1}]$, its order ideal is principal and $\Delta_K(t)$ is a generator of that ideal. This defines $\Delta_K(t)$ uniquely up to multiplication by an element of the form $\pm t^{\pm n}$, elements of this form being the units of the ring.

The Alexander polynomial of a knot can be defined as a unique element of $\mathbb{Z}[t, t^{-1}]$ if it is normalized so that $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) = 1$, and an analogous normalization can be made for links. This normalization was employed by Conway [6] in his famous paper on the enumeration of knots and links and the computation of link invariants where he developed the following idea (conceived, in an unnormalized form, by Alexander, [1] p. 301). Suppose that K_+, K_-, and K_0 are presentations, i.e. planar pictures, of three oriented links that are exactly the same except near one point where they are as in Fig. 1.

The normalized Alexander polynomial then satisfies the formula

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) + (t^{1/2} - t^{-1/2})\Delta_{K_0}(t) = 0.$$

Here $t^{1/2}$ is just a *formal* square root of t; if one writes $z = (t^{1/2} - t^{-1/2})$ then Δ_K can be

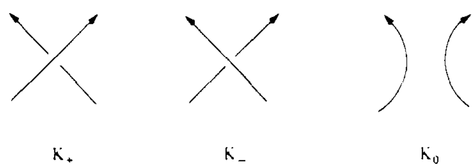
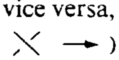


Fig. 1.

expressed as an element of $\mathbb{Z}[z]$ called the Conway potential function $\nabla_K(z)$. As presented in Conway [6], if one adds the information that the Alexander polynomial of the unknot, denoted $\mathscr{U}$, is 1 (and deduces that the polynomial for the trivial link, or 'unlink', of two or more unknots is zero) then the above formula allows the calculation of the polynomial for any oriented link. By a sequence of crossing changes any link can be changed to a trivial link for which the polynomial is known: assuming inductively that the polynomial is known for projections with fewer crossings, one applies the formula to the sequence of changes and calculates the required polynomial. Specific examples of this will be given here later. This method of calculation involves many arbitrary choices and appears to refer, in fundamental ways, to the specific presentation of links rather than to isotopy classes of S^1s in S^3. It was not until 1981 that it was shown, by Ball and Mehta [2], that this method of calculation could be made canonical with respect to a specific presentation of a knot or link and that the polynomial it produces is invariant under the Reidemeister moves (their proof capitalizes on the fact that the Conway potential contains no negative powers of z). Thus the above formula and the fact that $\Delta(t) = 1$ can be taken as axioms for the Alexander polynomial that imply existence, uniqueness and a ready method of calculation.

Recently a completely new Laurent polynomial invariant $V_K(t)$ for an oriented link K has been defined by Jones [12]. It has many properties that are rather similar to those of the Alexander polynomial and yet is by no means the same. Jones begins with K expressed as a closed braid so that it corresponds to an element α of the braid group on n strands, B_n, for some n. He then defines a representation, π, of B_n to the group of units of a certain (Hecke) algebra over the field of fractions of $\mathbb{Z}[t^{1/2}]$ on which is defined a trace function. He defines $V_K(t) = \mu^{n-1}$ trace$(\pi\alpha)$ where $\mu = -(t^{1/2} + t^{-1/2})$. By studying the structure of the braid group, conjugacy therein, and the Markov moves (see Birman [3]), Jones shows that $V_K(t)$ is indeed a link invariant. If one changes a crossing in the closed braid K some generator changes to its inverse, or vice versa, in the expression of K as a braid group word α, and if the crossing is removed, e.g.  , then the generator, or its inverse, is eliminated from the word α altogether. Thus in the context of closed braids it is easy to calculate any relationship that may exist between $V_{K_+}(t)$, $V_{K_-}(t)$ and $V_{K_0}(t)$ where K_+, K_-, and K_0 are closed braids related as in the previous paragraph. The relationship is

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) + (t^{-1/2} - t^{1/2})V_{K_0}(t) = 0.$$

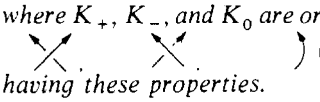
It is also true that $V_{\mathscr{U}}(t) = 1$ where $\mathscr{U}$ denotes the unknot. Thus, granting that $V_K(t)$ is well defined, this formula could be employed to calculate $V_K(t)$ for any link, just as in the case of the Alexander polynomial above.

The similarity between the formulae for $\Delta_K(t)$ and $V_K(t)$ is, of course, too great to be a unique coincidence; they are particular instances of a more general polynomial invariant for isotopy classes of oriented links which is a polynomial in two variables. This two-variable polynomial is the subject of this paper. Of course, in the very few months since Jones' surprising discovery, others have also considered how to generalize $V_K(t)$, and the following existence and uniqueness theorem was independently and simultaneously (August 1984) announced by Freyd and Yetter; Ocneanu; and Hoste; and also, independently (January 1985), by Przytycki and Traczyk [7]. This main theorem is as follows.

THEOREM. *To each oriented classical link* K *a unique element* $\mathscr{P}(K) = K(l, m) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ *can be associated so that* $K(l, m)$ *depends only on the isotopy class of* K; *if* $\mathscr{U}$ *is the unknot, then* $\mathscr{U}(l, m) = 1$; *and,*

$$lK_+(l, m) + l^{-1}K_-(l, m) + mK_0(l, m) = 0,$$

where K_+, K_-, *and* K_0 *are oriented links that are identical except near one point where they are*



respectively. Moreover, there is only one such association having these properties.

[Note that the formula is the general linear expression between polynomials for K_+, K_-, and K_0. The theory could equally be developed with a different taste in notation; for example

$$xK_+(x, y, z) + yK_-(x, y, z) + zK_0(x, y, z) = 0$$

will do just as well, but might erroneously suggest, at first glance, that there are three variables of interest.]

From the theorem it follows immediately that

$$\Delta_K(t) = K(\mathrm{i}, \mathrm{i}(t^{1/2} - t^{-1/2})) \qquad \text{and}$$

$$V_K(t) = K(\mathrm{i}t^{-1}, \mathrm{i}(t^{-1/2} - t^{1/2}))$$

where i is a formal square root of -1. The proof of the theorem given here is a completely "elementary" combinatorial proof assuming nothing but knowledge of the Reidemeister moves [17], so that, *inter alia*, this paper independently defines as invariants the polynomials of both Alexander and Jones (though it is very much indebted to them both). It defines the grander two-variable polynomial, $K(l, m)$, as an invariant of the oriented link, K, though it fails to shed much light on the conceptual questions of interpretation of the polynomial within the context of standard algebraic topology. A fundamental and, as yet, unanswered question asks if there exist nontrivial knots having trivial two-variable (or even Jones) polynomial. The resolution of this question could lead to significant conceptual progress.

Before reading the proof of the theorem any reader unfamiliar with Conway's computation procedure should use the theorem (with faith in its correctness) to make a few exploratory calculations. First consider the triple of Fig. 2. Both K_+ and K_- are copies of the unknot and so have 1 as their polynomial. Thus, $l \cdot 1 + l^{-1} \cdot 1 + mK_0(l, m) = 0$. Hence $K_0(l, m) = \mu$, where we define $\mu = -(l + l^{-1})m^{-1}$. Similarly, kinking one component and inducting on the number of components, one observes that the polynomial of the c-component unlink, which we shall denote by $\mathscr{U}^c$, is μ^{c-1}. The next example to consider is the simple *left-handed* link L of two components. The triple of Fig. 3 determines that

$$l^{-1}L(l, m) + l\mu + m \cdot 1 = 0$$

so that

$$L(l, m) = (l + l^3)m^{-1} - lm.$$

The *left-handed* trefoil T features in the triple of Fig. 4 so that

$$l^{-1}T(l, m) + l \cdot 1 + mL(l, m) = 0; \text{ and hence}$$

$$T(l, m) = -2l^2 - l^4 + l^2m^2.$$

K_0 K_- K_+

Fig. 2.

L

Fig. 3.

T

Fig. 4.

[Note that the left-handed natures of L and T are reflected in the positive powers of l in their polynomials.] In this way, by changing crossings in order to obtain a link of less complexity whose polynomial is already known, the polynomial of any link can be calculated. In practice the manipulation of symbols that this involves is arduous and easily gives rise to error, so that it is best to employ a computer in creating a list of polynomials. Such a list is given at the end of this paper.

The simple calculation given above for the trefoil knot, T, exemplifies the proof of the theorem. Crossings (one in this case) of a link K are changed to obtain a picture of the unlink of c components ($c = 1$ here) that is *ascending* when regarded as starting from selected base points (indicated by the dots in L and T) on each component. The polynomial of such an ascending link is *defined* to be μ^{c-1}. Assuming inductively that the polynomial is defined on link projections with fewer crossings, it can be calculated for the given K using the formula of the theorem. Copious checks have to be made: independence of the polynomial on choice of base points and of choice of crossing changes is fairly easy; but, to obtain an ascending link one has to have the components ordered (from the bottom upwards) and a delicate induction is needed to establish independence of that order. Along the way one has to check that the polynomial is unchanged by those Reidemeister moves that do not increase the number of crossings beyond the number of crossings under consideration.

Once the polynomial is known to be a well-defined isotopy invariant, it is characterized by the theorem and so can be explored therefrom. Two useful checks on calculations are (i) substituting $l = \mathrm{i}$ gives $\mu = 0$, and one retrieves the Conway potential with $m = \mathrm{i}z$, and (ii) substituting $m = -(l + l^{-1})$ gives $\mu = 1$ and always reduces the polynomial to 1.

The following properties of the two-variable polynomial will be deduced.

(1) The lowest power of m in $K(l,m)$ is equal to $1-c$, where c is the number of components of K and the powers of l and m are either all even or odd depending upon whether the number of components of K is odd or even, respectively.

(2) Reversing the orientation of every component of K leaves the polynomial unchanged.

(3) If $\bar{K}$ is the obverse (mirror image) of K, then $\bar{K}(l, m) = K(l^{-1}, m)$. Thus to be amphicheirial a link must have a polynomial which is symmetric in l and l^{-1}.

(4) The polynomial of the separated (or distant) union of K_1 and K_2 is $\mu K_1(l, m)K_2(l, m)$.

(5) If K_1 and K_2 are oriented links in S^3 let $K_1 \# K_2$ be the link formed by removing from (S^3, K_i) an unknotted ball pair, to obtain a pair (B^3, K_i^-), and identifying together the boundaries of these pairs in a manner consistent with all the orientations. In general this 'connected union' (or sum) is neither connected nor well defined. Nevertheless its polynomial is independent of how the (oriented) union is defined and is equal to the product of the polynomials associated to K_1 and K_2.

(6) $K(l, m)$ can be viewed as the most general "linear" skein invariant for links in the sense of Conway (see Giller [9]). This will be explained in detail later, but it implies that mutation of a link does not change the polynomial. Roughly, mutation consists of removing a 2-string tangle from K, rotating it through angle π, and replacing it. This accounts for the (disappointing) fact that the two 11-crossing knots with $\Delta_K(t) = 1$ have the same $K(l, m)$. However, the fact that $K(l, m)$ is highly nontrivial, namely

$$(2l^{-2} + 7 + 6l^2 + 2l^4) + (-3l^{-2} - 11 - 11l^2 - 3l^4)m^2 + (l^{-2} + 6 + 6l^2 + l^4)m^4 + (-1 - l^2)m^6,$$

shows that $K(l, m)$ is stronger than $\Delta_K(t)$, and that it depends on more than the infinite cyclic cover of a link complement. As the right and left trefoils have distinct polynomials, $K(l, m)$ depends on more than the fundamental group of the link complement.

(7) If A and B are 2-string tangles, let $A + B$, A^N, and A^D denote, respectively, the Conway sum, and the numerator and denominator of A. In Proposition 12 it is shown that $(1 - \mu^2)(A + B)^N = (A^N B^D + A^D B^N) - \mu(A^N B^N + A^D B^D)$.

(8) Using the formula of (7), or arguing directly, formulae will be deduced for the polynomials of rational (2-bridge) knots and links, and also for pretzel knots and links. Similar treatment can be given to the general arborescent link.

(9) There exist pairs of knots with the same polynomial but different genus, others with the same polynomial have different unknotting number, also there exist pairs with the same polynomial but different signature. Because skein equivalence *knots* (1-component links) have equal signatures, $K(l, m)$ does not capture skein equivalence completely.

(10) There is a knot, 11_{388} of Perko [16], which is not distinguishable from its obverse by either the Alexander or Jones polynomials but which is distinguished by the new polynomial.

An appendix to this paper lists the polynomials of oriented knots and links of low crossing number. A much more comprehensive listing will be published by M. B. Thistlethwaite in [21], where other link invariants will also be tabulated.

§1. THE EXISTENCE OF INVARIANT POLYNOMIALS

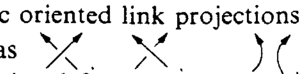
The goal of this section is to provide a completely elementary proof that there exists a two variable polynomial invariant for oriented links in $\mathbb{R}^3$, that is a proof which is based upon the "first principles" of a geometric analysis of oriented links by the study of generic projections to a plane. For the purposes of the definition of the polynomial we propose the following.

(1) A link is *ordered* if an order is given to its components.
(2) A link is *based* if a basepoint is specified on each component.
(3) A link is *oriented* if an orientation of each component is specified.
(4) A projection of a based link is *generic* if the projection defines an immersion of the link into the plane having no triple points and only transverse (and therefore finitely many)

double points such that the basepoints are distinct from the double points. Following normal procedure we preserve the under/over crossing information in the planar picture of a generic projection. Two projections are considered equivalent if they differ by an isotopy of the plane.

(5) The set of generic based ordered oriented link projections with at most n double points (or crossings) is denoted by $\mathscr{L}_n$. Let $\mathscr{L} = \bigcup_n \mathscr{L}_n$.

(6) An element L of $\mathscr{L}$ is said to be *ascending* if, when traversing the components of L in their given order and from their base points in the direction specified by their orientation, every crossing is first encountered as an under-crossing. Note that every ascending element is isotopic to the appropriate unlink. The exploitation of this fact provides one of the conceptual keystones of the method employed here.

(7) We employ the notation K_+, K_-, K_0 to identify generic oriented link projections which are identical outside a disk, inside which one has  respectively. These are normally described as right (or positive), left (or negative), and vacuous (or null) crossings respectively.

(8) We let $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ denote the ring of finite Laurent polynomials with integer coefficients in two variables l and m.

(9) The Reidemeister moves of types (i), (ii), and (iii) which do not increase the number of crossings are shown in Figs 5(i)–(iii).

The theorem to which this section is devoted is given next.

THEOREM. *There is a unique function $\mathscr{P}$ which associates to each $K \in \mathscr{L}$ an element $\mathscr{P}(K) = K(l, m) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which depends only on the isotopy class of the oriented link*

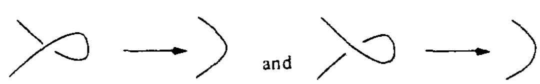


Fig. 5(i).

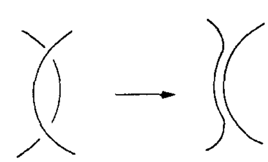


Fig. 5(ii).

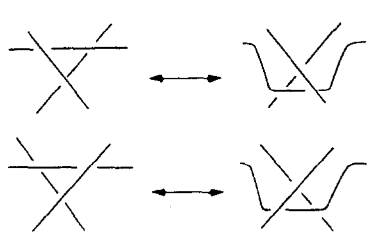


Fig. 5(iii).

and such that:

if K_+, K_-, K_0 are identical except for a single right, left and vacuous crossing, respectively, then

(I) $lK_+(l, m) + l^{-1}K_-(l, m) + mK_0(l, m) = 0$;

and if $\mathscr{U}$ denotes the unknot (of one component), then

(II) $\mathscr{U}(l, m) = 1$.

The proof of the theorem will be by induction on the number of crossings of a projection, will require a carefully chosen inductive assumption, and will concern the independence of the definition of the polynomial on the recursive method of its definition.

It is well known that a presentation of a knot can be changed to a presentation of the unknot by altering some of its crossings from overpasses to underpasses. A mild generalization of that idea *associates* to a generic projection, K, of an oriented, ordered, based link a *standard ascending* projection of the unlink of the same number of components, denoted $\alpha(K)$. This is obtained by starting at the basepoint of the first component and proceeding along that component changing (where necessary) overpasses to underpasses (and vice versa) so that every crossing is first encountered as an underpass. Continue from the basepoints of the second and all subsequent components in the same way changing crossings so that every crossing is first encountered as an undercrossing. This process geometrically separates and unknots the components thereby creating an unlink.

Inductive Hypothesis $(n-1)$. Assume that to each $K \in \mathscr{L}_{n-1}$, there is associated an element $\mathscr{P}(K)$ in $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which is independent of the choices of basepoints and the ordering of the components, is invariant under those Reidemeister moves which do not increase the number of crossings beyond $n-1$, and which satisfies formulae I and

(II′) if $\mathscr{U}^c \in \mathscr{L}_{n-1}$ denotes any ascending projection of c components and $\mu = -(l + l^{-1})/m$ then,

$$\mathscr{U}^c(l, m) = \mu^{c-1}.$$

The induction starts with zero crossing projections for which there is nothing to prove.

The Recursive Definition (n). If $K \in \mathscr{L}_n$ is any *standard* (oriented, ordered, based) *ascending* projection *define* $\mathscr{P}(K)(l, m)$ to be μ^{c-1} where c is the number of components of K. Otherwise, beginning at the basepoint of the first component of $K \in \mathscr{L}_n$ and proceeding in the direction specified by the orientation, change those crossings necessary so that each crossing is first encountered as an under-crossing. Continue the procedure with the remaining components in the sequence determined by the ordering, proceeding from the basepoint in the direction determined by the orientation, changing crossings so that ultimately every crossing is first encountered as an under-crossing. This results in the standard ascending projection $\alpha(K)$ associated to K. Employing formula I at each crossing change specified by the above unknotting algorithm (in the specified sequence), the Inductive Hypothesis $(n-1)$ applied to each K_0 (with arbitrary choices of basepoints and orders of components, which by the hypotheses are irrelevant to the value of each $\mathscr{P}(K_0)$), and *the definition* μ^{c-1} for the terminal situation $\alpha(K)$, calculate an integral polynomial in $\mathbb{Z}(l^{\pm 1}, m^{\pm 1}]$.

A priori, for elements of $\mathscr{L}_n$, this polynomial depends upon the specific sequence of crossing changes specified by the algorithm and hence the choice of basepoints, and the ordering of the components. Furthermore it might be changed under Reidemeister moves which do not increase the number of crossings, and it might not satisfy formula I. We shall prove a series of propositions to show that this is not the case and, thereby, prove the inductive hypothesis (n).

As indicated previously we shall employ the same symbol, K, for an element of $\mathscr{L}_n$ and for $\mathscr{P}(K) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, i.e. $\mathscr{P}(K)(l, m) \equiv K(l, m)$. We shall order the components of K by listing them sequentially $c_1, \ldots, c_s$ and shall label the crossings by a natural number $\{1, \ldots, n\}$. By a *segment* of the given projection we shall mean a component of the complement of the double points.

PROPOSITION 1 (n). *Suppose $K \in \mathscr{L}_n$. If the crossings of K that differ from those of $\alpha(K)$ are changed in* any sequence *to achieve $\alpha(K)$, then the corresponding calculation (using formulae* I *and* II′) yields $\mathscr{P}(K)$.

Proof. Inducting on the number of crossing differences between K and $\alpha(K)$, it is only necessary to consider altering the sequence by interchanging the first two crossing switches which the algorithm requires at, say, the crossing labelled i and then at the crossing labelled j. Let $\sigma_i K$ and $\eta_i K$ be the same as K except that the ith crossing is switched in $\sigma_i K$ and nullified in $\eta_i K$. Basepoints and component order of $\eta_i K$ are chosen arbitrarily, the choices having no effect on $P(\eta_i K)$ by the induction. Let ε_i be the sign of the ith crossing in K.

First consider the given sequence, σ_i before σ_j. To compute the polynomial we employ formula I,

$$\begin{aligned} K(l, m) &= -l^{-2\varepsilon_i}(\sigma_i K) - ml^{-\varepsilon_i}(\eta_i K) \\ &= -l^{-2\varepsilon_i}(-l^{-2\varepsilon_j}(\sigma_j \sigma_i K) - ml^{-\varepsilon_j}(\eta_j \sigma_i K)) - ml^{-\varepsilon_i}(\eta_i K) \\ &= l^{-2(\varepsilon_i + \varepsilon_j)}(\sigma_j \sigma_i K) + ml^{-2\varepsilon_i - \varepsilon_j}(\eta_j \sigma_i K) - ml^{-\varepsilon_i}(\eta_i K). \end{aligned}$$

Computing with the reverse order we find that we would have the quantity

$$l^{-2(\varepsilon_i + \varepsilon_j)}(\sigma_i \sigma_j K) + ml^{-2\varepsilon_j - \varepsilon_i}(\eta_i \sigma_j K) - ml^{-\varepsilon_j}(\eta_j K).$$

Since $\sigma_j \sigma_i = \sigma_i \sigma_j$ we see that the first two terms are equal. By the inductive hypothesis we may invoke formula I for $\eta_i K$ and $\eta_j K$ to find

$$(\eta_i K) = -l^{-2\varepsilon_j}(\sigma_j \eta_i K) - ml^{-\varepsilon_j}(\eta_j \eta_i K) \qquad \text{and}$$
$$(\eta_j K) = -l^{-2\varepsilon_i}(\sigma_i \eta_j K) - ml^{-\varepsilon_i}(\eta_i \eta_j K).$$

Substituting these expressions above and noting that $\sigma_j \eta_i = \eta_i \sigma_j$, $\sigma_i \eta_j = \eta_j \sigma_i$, and $\eta_i \eta_j = \eta_j \eta_i$ we see that the two expressions are equal.

In the next proposition we wish to show that the polynomial is independent of the choice of basepoints. It is at this point that we must make use of the specific value of μ to ensure this independence.

PROPOSITION 2(n). *$\mathscr{P}(K)$ is independent of the choice of basepoints.*

Proof. We need only show that if a basepoint of a component lies on a segment of the projection it can be moved to an adjacent segment of the component without changing the polynomial. Suppose the basepoint on component c_i is to be moved from position b_1 to position b_2, past a crossing of c_i with c_j (see Figs 6 and 7). Let K_1 and K_2 denote the relevant elements of $\mathscr{L}_n$ that have basepoints on c_i at b_1 and b_2, respectively, and are otherwise the same.

Case(a) $i \neq j$. In this case $\alpha(K_1) = \alpha(K_2)$ so $\mathscr{P}(K_1) = \mathscr{P}(K_2)$ as, by Proposition 1(n) the choice of sequence of crossing changes is irrelevant.

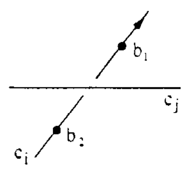


Fig. 6.

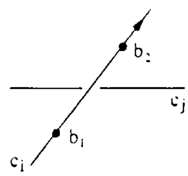


Fig. 7.

Case(*b*) $i = j$. In this case $\alpha(K_1)$ and $\alpha(K_2)$ differ only at the crossing under consideration, labelled r. By Proposition 1 (n), $\mathscr{P}(K_1)$ can be calculated by changing all the other relevant crossings first, giving $\mathscr{P}(K_1) = f(\mathscr{P}\sigma_r\alpha(K_1))$ where f is some (linear) function. Similarly $\mathscr{P}(K_2) = f(\mathscr{P}(\alpha(K_2)))$, the same function f occurring because the calculation involves only projections of fewer crossings for which (inductively) the position of basepoints is irrelevant. But, by definition $\mathscr{P}(\alpha(K_2)) = \mu^{c-1}$ (where K has c components) and further

$$l^{\varepsilon}\mathscr{P}(\sigma_r\alpha(K_1)) + l^{-\varepsilon}\mathscr{P}(\alpha(K_1)) + m\mathscr{P}(\eta_r\alpha(K_1)) = 0.$$

As $\mathscr{P}(\alpha(K_1)) = \mu^{c-1}$ and $\mathscr{P}(\eta_r\alpha(K_1)) = \mu^c$, (because $\eta_r\alpha(K_1) \in \mathscr{L}_{n-1}$ and $\eta_r\alpha(K_1)$ is an ascending configuration since, if one modifies the given order of components by inserting the two new components in the place of c_i with that containing $b_1(b_2)$ first in Fig. 6 (Fig. 7), each crossing is first encountered as an under-crossing) we deduce $\mathscr{P}(\sigma_r\alpha(K_1)) = \mu^{c-1}$. Thus, substituting in f, $\mathscr{P}(K_1) = \mathscr{P}(K_2)$.

PROPOSITION 3 (n). *$\mathscr{P}|\mathscr{L}_n$ satisfies formula* I.

Proof. Suppose K_+, K_-, and K_0 are in $\mathscr{L}_n$. The formula $l\mathscr{P}(K_+) + l^{-1}\mathscr{P}(K_-) + m\mathscr{P}(K_0) = 0$ is the first step in a calculation (permitted by Proposition 1(n)) of $\mathscr{P}(K_+)$ from $\mathscr{P}(\alpha K_+)$ *or* it is the first step in a calculation of $\mathscr{P}(K_-)$ from $\mathscr{P}(\alpha K_+)$ depending upon which of K_+ and K_- differs from $\alpha K_\pm$ at the crossing in question.

PROPOSITION 4 (n). *$\mathscr{P}(K)$ is invariant under Reidemeister moves which do not increase the number of crossings beyond n.*

Proof.

Type (i). We place the basepoint immediately before the crossing to be removed by one of the moves shown in Fig. 5(i), and a basepoint on the corresponding arc with that crossing removed (see Fig. 8 for an example). Note that the algorithm defines the same polynomial in each case.

Type (ii). In the case $i \leqq j$ we may place, in Fig. 9, the basepoint so that no crossing switch occurs in the initial configuration and, as a consequence, the polynomials which are computed by the algorithm are identical (using inductively this Reidemeister move on each relevant K_0).

Suppose now that $j < i$ so that we are obliged to change crossings labelled 1 and 2. We note that, independent of orientation, they are opposite crossings, ε being the sign of crossing 1, and compute that

$$\begin{aligned} K &= -l^{-2\varepsilon}(-l^{2\varepsilon}(\sigma_2\sigma_1K) - ml^{\varepsilon}(\eta_2\sigma_1K)) - ml^{-\varepsilon}(\eta_1K) \\ &= (\sigma_2\sigma_1K) + ml^{-\varepsilon}((\eta_2\sigma_1K) - (\eta_1K)). \end{aligned}$$

There are two cases to consider according as $\varepsilon = \pm 1$.

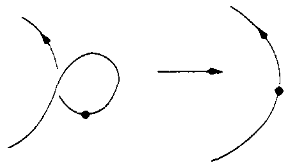


Fig. 8.

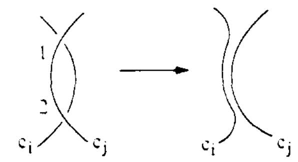


Fig. 9.

Case $\varepsilon = +1$. Here we find the situations in Fig. 10 which give identical polynomials.

Case $\varepsilon = -1$. By two applications of the invariance under Reidemeister moves (type (i)) we see that the projections of Fig. 11 give identical polynomials.

Thus we find that

$$K(l, m) = (\sigma_2\sigma_1 K)(l, m)$$

and for $\sigma_2\sigma_1 K$ we have the "$i \leqq j$" situation.

Type (iii). We shall first observe that, by virtue of Proposition 3(n) and the previous cases we may change the crossings between any pair of segments of the pictures that are at adjacent levels, in the *vertical* order of their appearance as presented in Fig. 12 (where the segment of c_k is shown above that of c_j which is above that of c_i), preserving the relationship between the polynomials before and after τ. Suppose, for example, that we wish to change the crossing of sign ε between components c_i and c_j in Fig. 12. We compute, using Proposition 3(n), that

$$K = -l^{-2\varepsilon}(\sigma K) - ml^{-\varepsilon}(\eta K) \text{ and}$$

$$(\tau K) = -l^{-2\varepsilon}(\sigma\tau K) - ml^{-\varepsilon}(\eta\tau K).$$

The observation follows by showing that $\mathscr{P}(\eta K)$ and $\mathscr{P}(\eta\tau K)$ are identical. As in case (ii) there are two cases depending upon ε. Fig. 13 shows the case when $\varepsilon = 1$, Fig. 14 shows the case $\varepsilon = -1$. In each case the two pairs of projections have the same polynomial; either they are the "same" projections (i.e. up to an isotopy of projections respecting the double points), or two applications of invariance under Reidemeister moves of type (ii) prove equality.

Thus, by changing the heights of segments in both K and τK to define K' and $\tau K'$, (without changing the relationship between the respective polynomials before and after τ), we can reduce the problem to an analysis of the case $i \leqq j \leqq k$ and such that, if we have equality, the segments in the support of τ are placed in ascending position. Under these conditions, the crossing changes that are required to evaluate $\mathscr{P}(K')$ and $\mathscr{P}(\tau K')$ are identical, hence their polynomials are identical (inductively using this Reidemeister move on each relevant K_0). Hence the polynomials for K and τK are identical.

In the next two propositions moves more general than those of Reidemeister will be discussed. The moves will be on a *non*-standard ascending element K of $\mathscr{L}_n$, that being a projection that is a standard ascending projection with respect to some ordering of the

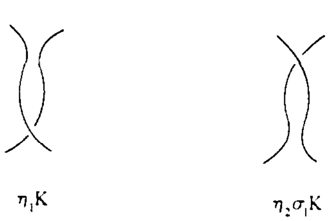


Fig. 10.

Fig. 11.

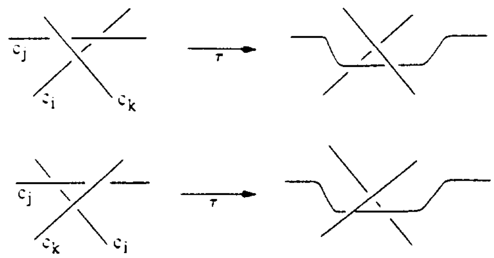


Fig. 12.

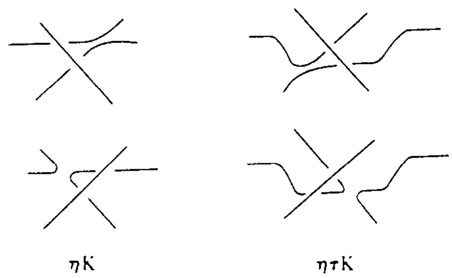


Fig. 13.

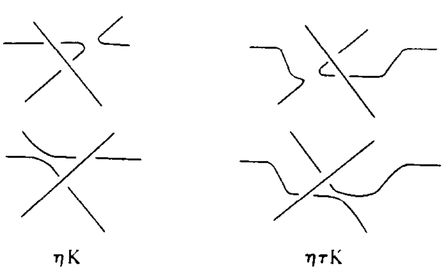


Fig. 14.

components other than the given ordering (and with some choice of basepoints and the given orientations). At this stage the polynomial of such an element is *not* apparent.

PROPOSITION 5(n) (*moving arcs*). *Suppose that K is a non-standard ascending element of* $\mathscr{L}_n$. *Let D be a disk in the projection plane such that* $D \cap K$ *is the union of an arc a in* ∂D *and a finite number of arcs* (to be called transversals) *properly embedded in D, an example of which is shown in Fig.* 15. *Suppose that no basepoint is in D, that each transversal crosses a in one point and that no pair of transversals cross in more than one point. Let b be the closure of* $\partial D - a$, *and let* $\hat{K}$ *be the result of substituting b for a in K, with b crossing over or under each transversal with the same choice as a. Then* $\mathscr{P}(K) = \mathscr{P}(\hat{K})$. (*Note that* $\hat{K}$ *is also ascending.*)

Proof. (Induction on the number, v, of transversals). The case $v = 0$ is trivial so we suppose that the proposition is true for $(v-1)$ transversals. Let N and S denote the endpoints of a. Choose *a* transversal arc, t, that is *northernmost* in the sense that there is no other transversal arc which meets both a and b nearer to N than does t. The part of the disk D lying north of t, less a very small neighbourhood of a, can be regarded as a disk as in the statement of the Proposition (with t now playing the rôle of a) with fewer than v transversal arcs. Thus we may use induction on v to more t, as shown in Fig. 16, without changing the polynomial.

Now use Proposition 4(n) to show that we may change the projection by Reidemeister moves of the third type along the shaded triangle (starting at $t \cap a$) so as to leave the polynomial unchanged and to eliminate the intersection of t and D. The fact that K is ascending ensures that at each usage of such a move the three arcs with which the move is concerned are indeed stacked one "above" the other as required for the move. The resulting situation is still ascending. This resulting configuration, as shown in Fig. 17, has fewer transversals and thus, by induction on v, we may replace a with b without changing the polynomial.

The reverse of the above process leaves the polynomial unchanged and restores t to its original position.

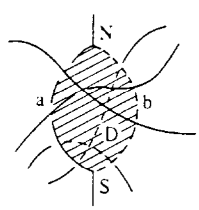


Fig. 15.

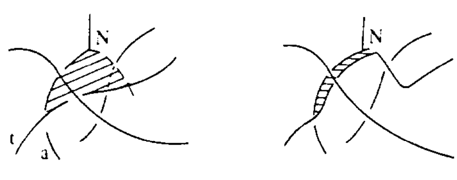


Fig. 16.

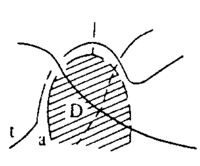


Fig. 17.

Corollary 5.1 (n). *Suppose that K and D are as in Proposition* 5(n). *Suppose, furthermore, that the transversals now have the properties that no two cross at more than one point, one transversal, denoted t, crosses a at two points, and each other transversal crosses each of a, t and b at one point. If, as before, $\hat{K}$ is the result of replacing a with b, then* $\mathscr{P}(K) = \mathscr{P}(\hat{K})$.

Proof. We first apply Proposition 5(n) to the case of $D' \subset D$, as shown in Fig. 19. Then apply Proposition 4(n) as shown in Fig. 20.

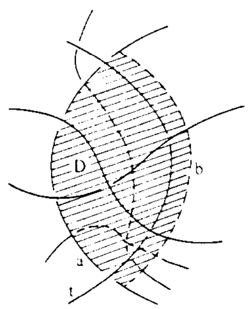


Fig. 18.

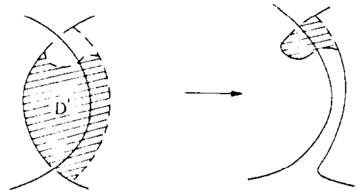


Fig. 19.

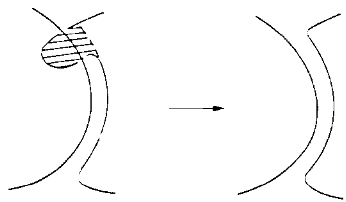


Fig. 20.

A *loop* in a link projection is a *simple* closed curve that is the projection of some sub-arc of the link (the loop then starts and ends at a double point of the projection) or the projection of an entire component (which thus has no self-crossing). Note that a loop in a link projection may contain many double points of the projection.

PROPOSITION 6(n). *The polynomial for $K \in \mathscr{L}_n$ is independent of the choice of order of the components.*

Proof. Let K have c components and n crossings, $n \geqq 1$. Without loss of generality the image in $\mathbb{R}^2$ of the projection of K is connected. This is because if $K = \cup_1^r K_i$ where the images in $\mathbb{R}^2$ of the K_i are disjoint, then $\mathscr{P}(K) = \mu^{r-1}\Pi_1^r(\mathscr{P}(K_i))$; the ordering on the K_i being induced by that of K.

By *definition* the polynomial of $\alpha(K)$, the standard ascending link associated to K with the given ordering and any choice of basepoints, is μ^{c-1}. Let $\alpha(K')$ be the standard ascending projection associated to K', the same geometric link with some other ordering of its components. Then give the components of $\alpha(K')$ the original order to give $\beta(K)$, a non-standard ascending projection. We calculate $\mathscr{P}(\beta(K))$ referring it, as we must by the algorithm, to $\alpha(K)$.

Choose an innermost loop of the projection of $\beta(K)$. If this loop contains no crossing of the projection (other than where the loop "starts" and "stops") it can be removed by a type (i) Reidemeister move without changing the polynomial (Proposition 4(n)). Thus $\beta(K)$ has the same polynomial as some other ascending element of $\mathscr{L}_{n-1}$, and for that element the ordering of components is, inductively, irrelevant. Hence $\mathscr{P}(\beta(K)) = \mu^{c-1}$. Otherwise there are transversals across the loop and, using Proposition 2(n) if necessary, it may be assumed that no basepoint lies inside the loop (nor on it unless the loop is the projection of a whole component). Thus within the loop there is an *innermost* occurrence of arcs a and t and disk D as in Corollary 5.1(n). Hence, using that corollary, a pair of crossings (of arcs a and t) can be removed, changing $\beta(K)$ to another ascending projection with the same polynomial and only $(n-2)$ crossings. As before the induction hypothesis implies that $\mathscr{P}(\beta(K)) = \mu^{c-1}$.

The calculation of the polynomial for the ordered oriented link K may be achieved by starting with the definition $\mathscr{P}(\alpha(K)) = \mu^{c-1}$, successively changing crossings in *any* sequence to change from $\alpha(K)$ to K, and invoking formula I and the inductive definition for $(n-1)$ crossings. One can, if one so wishes, choose a sequence of crossings that takes $\alpha(K)$ to $\beta(K)$ and then $\beta(K)$ to K. As explained above one would thus calculate $\mathscr{P}(\beta(K))$ and find it to be μ^{c-1}, and then proceed to calculate $\mathscr{P}(K)$ from that information. This final calculation is simply the calculation of $\mathscr{P}(K')$ from $\mathscr{P}(\alpha(K'))$. Thus $\mathscr{P}(K) = \mathscr{P}(K')$ proving that $\mathscr{P}(K)$ is independent of the choice of order of components.

Proof (of the Theorem). As a consequence of the propositions and their corollaries we have proved Inductive Hypothesis (n). Thus, by induction

$$\mathscr{P}: \mathscr{L} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$$

is defined and is an invariant of the isotopy class of the oriented link since every oriented link has a projection in some $\mathscr{L}_k \subset \mathscr{L}$ and any two projections of isotopic links are in some $\mathscr{L}_n$ and are equivalent by a finite sequence of Reidemeister moves that do not increase the number of crossings beyond n crossings. By Proposition 3(n), now true for all n, $\mathscr{P}(K)$ satisfies formula I.

Suppose that $\hat{\mathscr{P}}$ were another such function from $\mathscr{L}$ to $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which is different from $\mathscr{P}$. Then there is an element $K \in \mathscr{L}_k$, for the smallest such k, with $\mathscr{P}(K) \neq \hat{\mathscr{P}}(K)$. Since $\mathscr{P} = \hat{\mathscr{P}}: \mathscr{L}_{k-1} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ one may employ formula I to imply inequality in the case of an unlink. But by induction on the number of components, $\mathscr{P}$ and $\hat{\mathscr{P}}$ must be equal on unlinks in $\mathscr{L}_k$.

§2. GENERAL THEORY OF THE NEW POLYNOMIAL

The elementary properties of the new polynomial mentioned in the introduction will now be proved. Some of them are almost obvious, some are a little more obscure. The aim of this section is to introduce them, when they naturally occur, as part of the geometric exploitation of the basic formula that defines the polynomial in the main theorem. The relevant geometric idea is that of Conway's skein theory, implicit in his paper [6] and expounded and expanded in his lectures of 1978. In [9] Giller gives a discussion of the theory tuned, as was then necessary, to the Conway potential functions (or normalized Alexander polynomial). That theory can now be restructured for the two-variable polynomial produced here, it is a remarkably natural theory. Be warned that skein theory begins with frightening generality, but here, at least, it will quickly be particularized.

Definition. A *room* R is a compact 3-dimensional submanifold of S^3 on the boundary of which a finite set of points is given, each marked either "in" or "out". An *inhabitant* of R is a properly embedded smooth, compact, oriented 1-manifold in R, which meets the boundary precisely in the given set of points where its orientation agrees with the in and out designations. The *preskein* of R is the set of isotopy classes, keeping the *boundary fixed* during isotopies, of all inhabitants of R.

Two useful examples of rooms are P (prison), the 3-ball with the empty subset on its boundary, and Q (quad), the 3-ball with two in and two out points as shown, together with a specimen inhabitant, in Fig. 21. The inhabitants of P have no communication at all through the boundary of their room, and so the theory of the preskein P is that for usual oriented links up to isotopy.

For a room R, let $M(R)$ be the free module over $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ generated by (elements in one to one correspondence with) the preskein of R. Let $L(R)$, *the linearization of* R, be the quotient of $M(R)$ by the submodule generated by all elements of the form $ls_+ + l^{-1}s_- + ms_0$, where s_+, s_-, and s_0 are elements of the preskein that have representative inhabitants identical except near a point where they are  . Thus $L(R)$ is formal linear sums, of elements of the preskein of R, with the now familiar form of equalities imposed on the structure.

The first example to consider is the prison room P. As any link can be reduced to an unlink by changing crossings, $L(P)$ is clearly generated by such unlinks. But, in $L(P)$, such a link of c components is μ^{c-1} times the unknot. Hence $L(P)$ is generated by the unknot. Further, the main theorem of the last section has proved that $L(P)$ is *freely* generated by the (class of the) unknot $\mathscr{U}$; any generator K is *uniquely* expressible as $\mathscr{P}(K)\mathscr{U}$.

Any inhabitant of the room Q that consists of just two properly embedded arcs can be changed either **0** or ∞ as shown in Fig. 22. Thus $L(Q)$ is generated by **0** and ∞ and those two preskeins with extra unlinked, unknotted simple closed curves. However, if s is an inhabitant of Q and s, together with the "distant union" of the unlink of c components, is denoted by $s \sqcup \mathscr{U}^c$, then *in* $L(Q)$,

$$s \sqcup \mathscr{U}^c = \mu^c s.$$

The argument is the now familiar 'kinking' argument given for unlinks in the introduction. Hence $L(Q)$ is generated by **0** and ∞.

A house is now to be thought of as some edifice that contains rooms. The intuitive idea is that a house has wiring within its walls that permits communication between one room and another, and between rooms and the outside world.

Definition. A *house* H is a connected room together with a specific inhabitant (the *wiring*) and specified components $R_1, R_2, \ldots, R_n$ of $S^3 - H$ (called *the rooms of* H), where the input and output points on ∂R_i are the points of $\partial R_i \cap$ (wiring), with in and out designations determined by the orientation of the wiring. Let $\hat{H}$ be the room which contains the set H, and has boundary $\partial H - \bigcup_{i=1}^{n} \partial R_i$ on which the in and out points agree with those of H (thus $\hat{H}$ is H with its rooms "filled in").

PROPOSITION 7. *A house H with rooms $R_1, R_2, \ldots, R_n$ defines a multilinear map (in the category of modules)*

$$L(R_1) \times L(R_2) \times \cdots \times L(R_n) \to L(\hat{H}).$$

Proof. Insertion of an inhabitant into each room produces an inhabitant of $\hat{H}$. This function on generators extends by multilinearity to a multilinear function $M(R_1) \times M(R_2) \times \cdots \times M(R_n) \to M(\hat{H})$ which passes to quotients to give immediately the required function.

This proposition is important for its applicability rather than its erudition. The next result is a simple consequence.

Fig. 21.

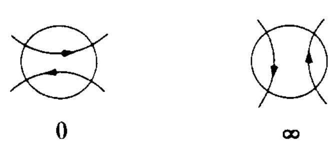


Fig. 22.

PROPOSITION 8. *If K_1 and K_2 are two links in S^3, separated by a 2-sphere, then*

$$\mathscr{P}(K_1 \sqcup K_2) = \mu \mathscr{P}(K_1)\mathscr{P}(K_2).$$

Note: It is common to write $K_1 \sqcup K_2$ for this distant union of K_1 and K_2.

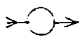
Proof. Let H be the house consisting of a ball with two sub-balls removed from its interior, and with empty wiring. Those two sub-balls are to be the two rooms of H. Into each of those rooms the preskein of P is inserted so that the previous proposition gives a bilinear map $L(P) \times L(P) \to L(P)$. But $L(P)$ is freely generated by $\mathscr{U}$, the class of the unknot, and, the above map sends $(\mathscr{U}, \mathscr{U})$ to $\mu\mathscr{U}$ since the image of $(\mathscr{U}, \mathscr{U})$ is the unlink of two components. Hence $(\mathscr{P}(K_1)\mathscr{U}, \mathscr{P}(K_2)\mathscr{U}) \to \mu\mathscr{P}(K_1)\mathscr{P}(K_2)\mathscr{U}$ by bilinearity, however, the right hand side is $\mathscr{P}(K_1 \sqcup K_2)\mathscr{U}$.

If K_1 and K_2 are oriented links in S^3 let $K_1 \# K_2$ be a link formed by removing from (S^3, K_i) an unknotted ball pair, to obtain a pair (B^3, K_i^-), and identifying the boundaries of these pairs in a manner consistent with all orientations. In general this "connected sum" $K_1 \# K_2$ is neither connected (as a subset of S^3) nor well defined. However, its polynomial is well defined.

PROPOSITION 9. *If $K_1 \# K_2$ is any connected sum of oriented links K_1 and K_2 in S^3, then*

$$\mathscr{P}(K_1 \# K_2) = \mathscr{P}(K_1)\mathscr{P}(K_2).$$

Proof. Let H be the house with two rooms of Fig. 24.

Each room is a copy of  with generator $g_i =$  for $L(R_i)$. The bilinear map $L(R_1) \times L(R_2) \to L(P)$ defined in Proposition 7 sends (g_1, g_2) to $\mathscr{U}$. It maps (K_1^-, g_2) *to* $\mathscr{P}(K_1)\mathscr{U}$ so $K_1^- = \mathscr{P}(K_1)g_1$, and similarly for K_2^-. Thus $(K_1^-, K_2^-) \mapsto \mathscr{P}(K_1)\mathscr{P}(K_2)\mathscr{U}$; but inserting K_1^- into R_1 and K_2^- into R_2 produces the link $K_1 \# K_2$, so that $(K_1^-, K_2^-) \to \mathscr{P}(K_1 \# K_2)\mathscr{U}$.

This result provides an elementary method of obtaining distinct links with the same polynomial. For example, if K_1, K_2 and K_3 are distinct *knots*, then $(K_1 \# K_2) \sqcup K_3$ and $K_1 \sqcup (K_2 \# K_3)$ both have polynomial $\mu\mathscr{P}(K_1)\mathscr{P}(K_2)\mathscr{P}(K_3)$.

Definition. Let K be an oriented link in (oriented) S^3. The *reverse* of K is the same link but with the orientation of each component of the sub-1-manifold of S^3 changed. The *obverse* of K is the same link as K but with the orientation of S^3 changed.

Thus in a given presentation of K as a diagram with arrows on the components and over-crossings marked, the reverse of K, *rev* K, is obtained by changing all the arrows, the obverse of K, $\bar{K}$, is obtained by changing all over-crossings to under-crossings (since the orientation of $S^3 = \mathbb{R}^3 \cup \infty$ can be reversed by taking the reflection of S^3 in the plane of projection of the link, thereby reversing all the crossings of the given projection). The next result states how the polynomial is influenced by these two operations on links. It is easy to deduce this result

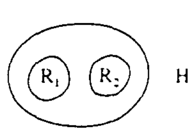


Fig. 23.

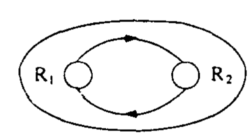


Fig. 24.

directly from the main theorem; here a proof using linear skein theory is given.

PROPOSITION 10. *Let K be an oriented link in (oriented) S^3.*

(i) $\mathscr{P}(\text{rev } K) = \mathscr{P}(K)$

(ii) $\mathscr{P}(\bar{K})(l, m) = \mathscr{P}(K)(l^{-1}, m)$.

Proof. (i) Consider the trivial room P. The operation 'rev' maps inhabitants of P to inhabitants of P and induces a linear map $L(P) \to L(P)$ because 'rev' does not change the concepts of positive and negative crossing in a presentation. However, rev $\mathscr{U} = \mathscr{U}$, so this linear map is the identity.

(ii) The map $K \to \bar{K}$, treated in the same way, induces a map $L(P) \to L(P)$ that is fixed on $\mathscr{U}$. However it exchanges the ideas of positive and negative crossing, so that this map is *semi-linear* with respect to the involution on $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ that is fixed on m and interchanges l and l^{-1}.

It follows that, in determining the basic symmetries of knots, the two-variable polynomial is useless on the question of reversibility. In fact if K is one of the knots of Trotter [22] that are non-reversible (Trotter uses 'non-invertible') then K and rev (K) form a pair of distinct knots with the same polynomial. A link is called *amphicheiral* if $K = \bar{K}$, and for this to occur Proposition 10 requires that $\mathscr{P}(K)(l, m) = \mathscr{P}(K)(l^{-1}, m)$. A glance at the tables at the end of this paper shows that this provides a good test for amphicheirality; it is not infallible since the knot 9_{42} has self-conjugate polynomial $(-2l^{-2} - 3 - 2l^2) + (l^{-2} + 4 + l^2)m^2 - m^4$ but, having non-zero signature, it cannot be amphicheiral.

Linear skein theory and the ideas of rooms and houses would hardly be justified by the preceding discussion. Consideration of room Q is more significant. In fact it is convenient to generalize Q a little and let $\tilde{Q}$ denote any *one* of the rooms associated to the diagram of Fig. 25 with two inputs and two outputs allocated in any of the six possible ways.

Definition. Let K_1 and K_2 be oriented links in S^3, then K_2 is a *mutation* of K_1 (and vice versa), if K_2 can be obtained from K_1 by the following process:

(i) remove from K_1 an inhabitant T of a copy of $\tilde{Q}$;

(ii) rotate T through angle π about the central axis (perpendicular to the plane of the diagram) *or* about the E–W *or* the N–S axis *and* if necessary change *all* the arrows to achieve another inhabitant of $\tilde{Q}$;

(iii) place this new inhabitant in $\tilde{Q}$ to obtain K_2.

PROPOSITION 11. *If K_1 and K_2 are oriented links in S^3 and K_2 is a mutation of K_1, then* $\mathscr{P}(K_1) = \mathscr{P}(K_2)$.

Example 11.1 The Kinoshita–Terasaka knot and the Conway knot (see Fig. 26) have the same polynomial. Note that by the work of Gabai [8] these are knots of different genus.

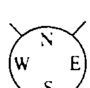
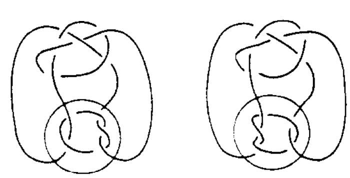


Fig. 25.

C



K − T

Fig. 26.

Proof. (of Proposition 11) Let ρ be one of the three involutions on the set of inhabitants of $\tilde{Q}$ described in the above definition, ρ being a rotation through π possibly followed by arrow reversal. Clearly ρ induces a linear map $\rho: L(\tilde{Q}) \to L(\tilde{Q})$. But, $L(\tilde{Q})$, with $\tilde{Q}$ denoting any *one* of the six possible rooms according to the specific choice of inputs and outputs, is generated by two of twelve tangles, (the choice of which of these depends upon the specific choice of inputs and outputs defining $\tilde{Q}$), namely those shown in Fig. 27, where each diagram has four choices of arrows. This has already been discussed in detail for the particular case $\tilde{Q} = Q$. However each of these is invariant under ρ so that ρ is the identity map on $L(\tilde{Q})$. From the definition of mutation there is a house H, a ball with one interior ball removed to give a room $\tilde{Q}$, such that insertion of T into $\tilde{Q}$ produces K_1, insertion of ρT produces K_2. But H induces, by Proposition 7, a linear map $L(\tilde{Q}) \to L(P)$ which sends T to $\mathscr{P}(K_1)$, and ρT to $\mathscr{P}(K_2)$. This proves the result since ρ is the identity on $L(\tilde{Q})$.

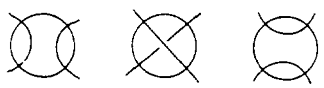


Fig. 27.

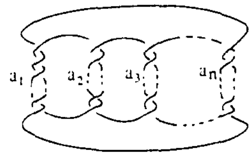


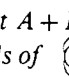
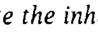
Fig. 28.

Example 11.2 In Fig. 28 is shown a pretzel link that will be denoted $L[a_1^{\varepsilon(1)}, a_2^{\varepsilon(2)}, \ldots, a_n^{\varepsilon(n)}]$. The ith "vertical" strip has a_i half twists, these being in a right-hand sense if a_i is positive, and in a left-hand sense if a_i is negative. The superscript $\varepsilon(i)$ is 1 if all the crossings on the ith strip are positive, and -1 if they are negative. Note that $\varepsilon(i)$ depends on the choice of orientation of the various components; for a given $(a_1, a_2, \ldots a_n)$ an arbitrary choice of ε may not be possible. Now, Proposition 11 implies that for any permutation $\sigma \in S_n$,

$$\mathscr{P}(L[a_1^{\varepsilon(1)}, a_2^{\varepsilon(2)}, \ldots a_n^{\varepsilon(n)}]) = \mathscr{P}(L[a_{\sigma 1}^{\varepsilon(\sigma 1)}, a_{\sigma 2}^{\varepsilon(\sigma 2)}, \ldots a_{\sigma n}^{\varepsilon(\sigma n)}]).$$

Thus, taking care concerning orientation, the polynomial is unchanged by permutation of the a_i. This follows from the proposition since, by mutation, the ith and $(i+1)$th strips can be interchanged leaving the polynomial fixed. Note that if all the a_i are odd, one can choose $\varepsilon(i)$ to be the opposite sign to that of a_i, and, in that limited way simplify the notation to $L[a_1, \ldots, a_n]$.

Next, the room Q is used to determine the polynomial of the total sum of two tangles discussed in [14]. This is the two-variable analogue of the numerator–denominator formula of Conway to which the next proposition reduces on substitution of $\mu = 0$.

PROPOSITION 12. *Let A and B be inhabitants of Q (in [14] these are essentially two-string tangles with extra closed loops), let $A + B$ denote the inhabitant of Fig. 29 and let A^N and A^D denote the two-variable polynomials of*  *and*  *respectively. Then $(\mu^2 - 1)(A + B)^N = \mu(A^N B^N + A^D B^D) - (A^N B^D + A^D B^N)$.*

Proof. By Proposition 7 a bilinear map $L(Q) \times L(Q) \to L(P)$ is defined by the house with rooms R_1 and R_2 of Fig. 30. With respect to bases, $\mathbf{0}$, ∞ for $L(Q)$ and $\mathscr{U}$ for $L(P)$ this is clearly represented by the matrix

$$\begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}.$$

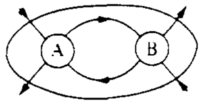


Fig. 29.

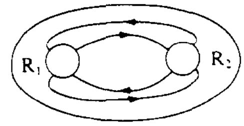


Fig. 30.

So, letting $(\alpha_0, \alpha_\infty)$ and (β_0, β_∞) be coordinates for A and B in $L(Q)$, $(A+B)^N$ is

$$(\alpha_0, \alpha_\infty)\begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_\infty \end{pmatrix}.$$

Taking values (1, 0) and then (0, 1) for (β_0, β_∞) gives

$$(A^N, A^D) = (\alpha_0, \alpha_\infty)\begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}$$

and similarly for B. Hence

$$(A+B)^N = (A^N, A^D)\begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}^{-1}\begin{pmatrix} B^N \\ B^D \end{pmatrix}$$

or

$$(\mu^2-1)(A+B)^N = (A^N, A^D)\begin{pmatrix} \mu & -1 \\ -1 & \mu \end{pmatrix}\begin{pmatrix} B^N \\ B^D \end{pmatrix}$$

which is the required answer.

Note: For ease of manipulation in the proof a temporary step was taken into the field of fractions of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$. One can, in the same way, establish the formula $(A+B)^D = A^D B^D$ but that is simply a restatement of Proposition 9. In the above proof it transpires that

$$(\mu^2-1)(\alpha_0, \alpha_\infty) = (A^N, A^D)\begin{pmatrix} \mu & -1 \\ -1 & \mu \end{pmatrix}.$$

Thus A^N and A^D determine α_0 and α_∞ *uniquely*, so **0** and ∞ form a *free base* for $L(Q)$. The polynomial of an arborescent knot or link (algebraic in the sense of [6]) can be regarded as $(A+B)^N$ where B is a rational tangle and A^N and A^D are the polynomials of arborescent links of a lower complexity. Thus Proposition 12 provides recursive formulae for finding the polynomial of such arborescent links; often however they seem unattractive in practice. The next proposition is a worked example for pretzel links with odd coefficients. Pretzel links are, of course, special Montesinos links, and hence arborescent.

In this next proposition a value for the polynomial of the r-crossing link of Fig. 31 will be needed. Let $\mathscr{P}[r]$ be this polynomial, $\mathscr{P}[0] = \mu$, $\mathscr{P}[1] = 1$. Changing one of the crossings produces the formula

$$\mathscr{P}[r] + lm\mathscr{P}[r-1] + l^2\mathscr{P}[r-2] = 0$$

so that

$$\begin{pmatrix} \mathscr{P}[r] \\ \mathscr{P}[r-1] \end{pmatrix} = \begin{pmatrix} -lm & -l^2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathscr{P}[r-1] \\ \mathscr{P}[r-2] \end{pmatrix}.$$

Hence

$$\mathscr{P}[r] = (1, 0)\begin{pmatrix} \mathscr{P}[r] \\ \mathscr{P}[r-1] \end{pmatrix}$$

$$= (1, 0)\begin{pmatrix} -lm & -l^2 \\ 1 & 0 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

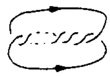


Fig. 31.

PROPOSITION 13. *Let K be the pretzel link* $L[a_1, a_2, \ldots, a_n]$, *where the* a_i *are odd integers and orientations are as described above* (Example 11.2). *Let* $\mathscr{P}[0] = \mu$ *and* $\mathscr{P}[r]$ *be*

$$(1,0)\begin{pmatrix} -lm & -l^2 \\ 1 & 0 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

Then $\mathscr{P}(K) = \sum_{\delta} x_{a_1}^{\delta(1)} x_{a_2}^{\delta(2)} \ldots x_{a_n}^{\delta(n)} \mathscr{P}\left[\sum_{i=1}^{n} \delta(i)\right]$,

where the summation is over the 2^n *functions* $\delta\colon \{1, 2, \ldots, n\} \to \{0, 1\}$, *and*

$$x_a^1 = (-1)^{\frac{a-1}{2}} l^{(a-1)},$$
$$x_a^0 = \mu^{-1}(1 - x_a^1).$$

Proof. Let $\tilde{Q}$ be the room depicted with inhabitants $\mathbf{x}_0$, $\mathbf{x}_1$ and T_a in Fig. 32 where T_a has a crossings. $L(\tilde{Q})$ is generated by $\mathbf{x}_0$ and $\mathbf{x}_1$. In $L(\tilde{Q})$, $l^{-1}T_a + lT_{a-2} + m\mathbf{x}_0 = 0$, and the solution for this is

$$T_a = x_a^0 \mathbf{x}_0 + x_a^1 \mathbf{x}_1.$$

The house with n rooms of Fig. 33 defines an n-linear map $\phi\colon (L(\tilde{Q}))^n \to L(P)$. Then

$$\mathscr{P}(K) = \phi(T_{a_1}, T_{a_2}, \ldots, T_{a_n})$$
$$= \phi(x_{a_1}^0 \mathbf{x}_0 + x_{a_1}^1 \mathbf{x}_1, \ldots, x_{a_n}^0 \mathbf{x}_0 + x_{a_n}^1 \mathbf{x}_1).$$

Now $\phi(z_1, z_2, \ldots z_n)$ where r of the z_i are $\mathbf{x}_1$ and the remaining $(n-r)$ are $\mathbf{x}_0$ is $\mathscr{P}[r]\mathscr{U}$. The result now follows using the multilinearity of ϕ.

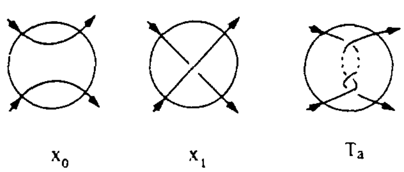


Fig. 32.

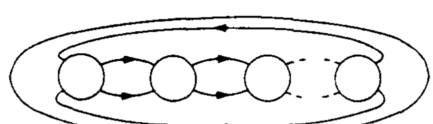


Fig. 33.

One notes that the expression for $\mathscr{P}(K)$ depends only on the set $\{a_1, a_2, \ldots, a_n\}$ of odd numbers and not upon their ordering as was expected from Example 11.2. The same general idea can be used when some of the a_i are even.

Seifert's famous pretzel knot with unit Alexander polynomial is $L[-3, 5, 7]$ though in fact $L[a_1, a_2, a_3]$ with the a_i odd has such an Alexander polynomial if $a_1a_2 + a_2a_3 + a_3a_1 = -1$. Substituting

$$x_{-3}^0 = \mu^{-1}(1 - l^4) \qquad x_{-3}^1 = l^{-4}$$
$$x_5^0 = \mu^{-1}(1 - l^4) \qquad x_5^1 = l^4$$
$$x_7^0 = \mu^{-1}(1 + l^6) \qquad x_7^1 = -l^6,$$

one obtains

$$\mathscr{P}(L[-3,5,7]) = (-l^2 + l^4 - l^8 + l^{10} + l^{12}) + (l^2 - l^4 + l^8 - l^{10})m^2.$$

Note this reduces to 1 if the substitution $l = i$ is made.

The next result uses the preceding general ideas in a slightly different way to find the polynomial of a rational (or 2-bridge) link.

PROPOSITION 14. *Let K be the rational knot or link $c_1 c_2 c_3 \ldots c_n$ in the notation of* [6], *where the c_i are even integers. For any integer r, let $M(2r) \in GL_2(\mathbb{Z}[l^{\pm 1}, m^{\pm 1}])$ be the matrix*

$$\begin{pmatrix} (1-(-1)^r l^{-2r})\mu^{-1} & (-1)^r l^{-2r} \\ 1 & 0 \end{pmatrix}.$$

Then $\mathscr{P}(K) = (1,0)\dot{M}(c_1) \ldots \bar{M}(c_{n-2}) M(c_{n-1}) \bar{M}(c_n) \begin{pmatrix} 1 \\ \mu \end{pmatrix}$, where the bar denotes conjugation with respect to interchanging l and l^{-1}; the conjugate of $M(c_r)$ occurs in the formula when $r \equiv n \bmod 2$.

Proof. First a word about notation is in order. The link K in question is that corresponding to the rational number p/q where p/q is given by the continued fraction

$$c_n + \cfrac{1}{c_{n-1} + \cdots \cfrac{1}{c_2 + \cfrac{1}{c_1}}}.$$

Here the unique expression with the c_i all even is required. Then K is the unique link [11] with the lens space $L_{p,q}$ as its double branched cover. In the notation of Proposition 12, $\mathscr{P}(K)$ is $T(c_1, c_2, \ldots, c_n)^N$ where, depending upon the parity of n, $T(c_1, c_2, \ldots, c_n)$ is one of the inhabitants of Q shown in Fig. 34, where each c_i denotes the number of crossings depicted in the sense shown (and ∂Q is omitted for simplicity). As already remarked, the module $L(Q)$ is generated by the preskeins $\mathbf{0}$ and ∞, so it must be possible in $L(Q)$ to express $T(c_1, \ldots c_n)$ in terms of these generators. A simple recurrence formula, analogous to that used in Proposition 12 shows that, in $L(Q)$, the preskeins X_{2r} and Y_{2r} of Fig. 35 are given by

$$X_{2r} = \alpha_{2r}\mathbf{0} + \beta_{2r}\infty$$

$$Y_{2r} = \bar{\beta}_{2r}\mathbf{0} + \bar{\alpha}_{2r}\infty$$

where

$$\alpha_{2r} = (1-(-1)^r l^{-2r})\mu^{-1}$$

$$\beta_{2r} = (-1)^r l^{-2r}.$$

Consider, in the case when n is even, the house with one room shown in Fig. 36. Let $\phi : L(Q) \to L(Q)$ be the associated linear map. Then

$$T(c_1, c_2, \ldots c_n) = \phi(X_{c_1}) = \alpha_{c_1}(\phi(\mathbf{0}) + \beta_{c_1}\phi(\infty).$$

Thus $T(c_1, c_2, \ldots c_n) = \alpha_{c_1} T(c_2, c_3, \ldots c_n) + \beta_{c_1} T(c_3, c_4, \ldots, c_n)$, which produces the

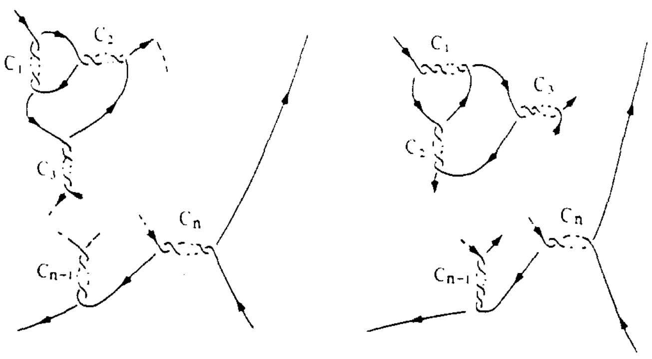


Fig. 34.

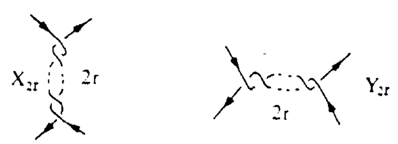


Fig. 35.

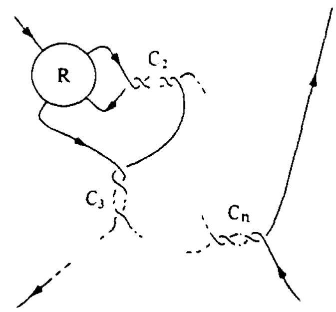


Fig. 36.

matrix equation

$$\begin{pmatrix} T(c_1, c_2, \ldots c_n) \\ T(c_2, c_3, \ldots c_n) \end{pmatrix} = M(c_1) \begin{pmatrix} T(c_2, c_3, \ldots c_n) \\ T(c_3, c_4, \ldots c_n) \end{pmatrix}.$$

For n odd the same equation with $\bar{M}(c_1)$ in place of $M(c_1)$ comes from an exactly similar method. Giving a proper interpretation to the symbols when $n = 1$ or 2, and iterating, one obtains

$$T(c_1, c_2, \ldots c_n) = (1, 0) \begin{pmatrix} T(c_1, c_2, \ldots c_n) \\ T(c_2, c_3, \ldots c_n) \end{pmatrix}$$

$$= (1, 0)\dot{M}(c_1) \ldots \bar{M}(c_{n-2})M(c_{n-1})\bar{M}(c_n)\begin{pmatrix} \infty \\ 0 \end{pmatrix}.$$

Here the dots over $M(c_1)$ indicate that conjugation takes place when n is odd. Now, as in Proposition 12, if T is an inhabitant of Q, and $T = \alpha 0 + \beta\infty$ in $L(Q)$, then $T^N = \mu\alpha + \beta$, so the required formula for $T(c_1, c_2, \ldots c_n)$ follows at once.

If the substitution $l = i$, $iz = m$ is made one obtains

$$\mathscr{P}(K)(i, iz) = \nabla_K(z)$$

where $\nabla_K(z)$ is the Conway potential of K (where $z = \{r\}$ in the notation of [6]). With this substitution $\mu = 0$, and $M(2r)$ and its conjugate become

$$\begin{pmatrix} -rz & 1 \\ 1 & 0 \end{pmatrix}$$

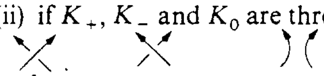
and $\nabla_K(z) = (1, 0)M(c_1)M(c_2) \dots M(c_n)\begin{pmatrix}1\\0\end{pmatrix}$, regaining the potential formula for rational links (see [9] for example).

In the proof of Proposition 14 a formula evolved expressing, in $L(Q)$, $T(c_1, c_2, \dots c_n)$ in terms of $\mathbf{0}$ and ∞ when all the c_i are even. In principle this formula can be utilized, in conjunction with the technique of Proposition 13 to obtain formulae for the polynomials of arborescent links, or to express the tangle, that is the characteristic arborescent part of a link [5] in terms of generators of the linearization of its associated room. That is of theoretical importance, but the anticipated complication of a general formula is unattractive.

Most of the preceding discussion of this section has been based on *linearized* skein theory applied to the preskeins of inhabitants of various rather simple rooms, the main useful idea being Proposition 7. A routine extension can be made to a room R with n inputs and n outputs as in [9], then $L(R)$ has a set of $n!$ generators, but details become complicated. It is now in order to mention *skein theory* proper, at least when applied to (the preskein of) oriented links in S^3. Skein theory germinated in [6] and was explained in [9]. The definitions of the theory tend to evolve, and the present situation is given below; it may be that the technique of proof of the main theorem in §1 will eventually induce change in those definitions. Skein theory is simply the study of equivalence classes of oriented links in S^3 under skein equivalence; a skein invariant is simply a function well defined on these equivalence classes.

Definition. Skein equivalence is the smallest equivalence relation "$\sim$" on the set of all oriented links in S^3 such that

(i) if K and L are ambient isotopic then $K \sim L$;

(ii) if K_+, K_- and K_0 are three oriented links identical except near one point where they are  respectively, and L_+, L_- and L_0 is another such (skein) triple then

(a) $K_+ \sim L_+$ *and* $K_0 \sim L_0$ *implies* $K_- \sim L_-$, and
(b) $K_- \sim L_-$ *and* $K_0 \sim L_0$ *implies* $K_+ \sim L_+$.

Thus skein equivalence is, in a sense, the minimal equivalence relation for which the proof of §1 will give a well defined skein invariant. The new two-variable polynomial can be regarded as the most general *linear* skein invariant.

PROPOSITION 15. *$\mathscr{P}(K)$ is a skein invariant (hence so are the polynomials of Alexander* [9] *and Jones* [12]).

Proof. This follows from the Theorem. (Signatures of *knots* are skein invariants and folklore asserts that Minkowski units are also.)

It is a pleasing exercise to check that any oriented link is skein equivalent to its reverse. Simply induct on the number of crossings and on the number of crossing changes necessary to create an ascending presentation. Similarly any two mutants are skein equivalent, the induction being on the number of crossings in the tangle to be 'rotated'. In fact this gives another way of viewing the proofs of Proposition 10(i) and of Proposition 11.

Amongst knots, skein equivalence *seems* to be a rare phenomenon except for iterated mutations. Of course, mutations of significance do not occur in knots of less than 11 crossings, but even for higher numbers of crossings it seems that a pair of knots is 'usually' distinguished by the two-variable polynomials and hence they are not skein equivalent.

Conversely one may search for skein equivalence by inspecting the polynomials.

Using a computer, Thistlethwaite has shown that amongst the 12,965 knots with at most thirteen crossings there are thirty with $\Delta(t) = 9 - 6(t + t^{-1}) + 2(t^2 + t^{-2})$. Examination of these failed to find a pair of knots distinguished by $\mathscr{P}(K(l, m))$ but not by $V_K(t)$ (*but* see Example 19). However, an outcome of that search produced the following extraordinary example.

Example 16. Figure 37 depicts three knots. These are all slice knots and so have zero signatures. That they are distinct is proved by Thistlethwaite's enumeration of representations of their knot groups into permutation groups. Now changing the encircled crossing of 13_{6714} produces 10_{129}, and nullifying that crossing produces $\mathscr{U}^2$, the trivial link of two components. Similarly, changing the encircled crossing in 10_{129} gives 8_8 and nullifying it gives $\mathscr{U}^2$. Hence we have triples $(13_{6714}, 10_{129}, \mathscr{U}^2)$ and $(8_8, 10_{129}, \mathscr{U}^2)$ both of the form (K_+, K_-, K_0). Thus 8_8 and 13_{6714} are skein equivalent; they both have polynomial

$$(-l^{-4} - l^{-2} + 2 + l^2) + (l^{-4} + 2l^{-2} - 2 - l^2)m^2 + (-l^2 + 1)m^4.$$

Now 8_8 is a rational knot with double branched cover the lens space $L_{25,11}$. Mutation of knots leaves double branched covers unchanged, and by [11] 8_8 is the only knot with $L_{25,11}$ as its double branched cover. Thus 8_8 has no mutants, and the above skein equivalence is distinct from the mutant idea. This example also shows that skein equivalent knots can have distinct double branched covers.

Incredibly, because of the almost-symmetry of $\mathscr{P}(8_8)$, $\mathscr{P}(10_{129}) = \overline{\mathscr{P}(8_8)}$, where the bar denotes exchanging l and l^{-1}. Then indeed $X = \mathscr{P}(8_8)$ satisfies $lX + l^{-1}\bar{X} + m\mu = 0$, an equation also satisfied by $X = 1$! Are 8_8 and the obverse of 10_{129} skein equivalent?

Before leaving this example consider unknotting numbers. The knot 10_{129} obviously has unknotting number 1, hence so does its obverse. However 8_8 has unknotting number 2. Clearly two crossing changes suffice to unknot 8_8. That two changes are necessary follows from consideration of the linking form λ on the first homology of the double branched cover (see [15]). $H_1(L_{25,11}) = \mathbb{Z}_{25}$ with generator g such that $\lambda(g, g) = 11/25$ in $\mathbb{Q}/\mathbb{Z}$. If 8_8 had unknotting number 1 there would be another generator tg such that $\lambda(tg, tg) = \pm 2/25$. Then $2t^2/25 = \pm 11/25$ in $\mathbb{Q}/\mathbb{Z}$ or $2t^2 \equiv \pm 11$ modulo 25. However no such integer t exists. Thus 8_8 and the obverse of 10_{129} have the same polynomial but distinct unknotting numbers. Because the determination of the two-variable polynomial is concerned with crossing changes it is reasonable to hope that it might give some information on unknotting numbers. This, not unexpected, example shows that complete unknotting number information cannot be obtained.

The next example, due essentially to Birman [4] shows that links may have the same polynomial but different signatures.

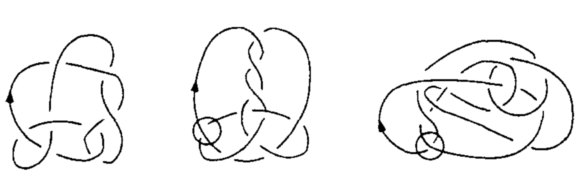


Fig. 37.

Example 17. The two knots depicted in Fig. 38 have the same polynomial. The common polynomial is

$$(-4l^6 - 3l^8) + (10l^6 + 4l^8)m^2 + (-6l^6 - l^8)m^4 + l^6m^6.$$

However according to [4] these knots have distinct signatures. Thus the value of the signature of a knot is not carried by the polynomial. Birman [4] gives a collection of pairs analogous to this example.

Since mutation is a prime example of skein equivalence, and mutants have the same double branched cover, it is well to record the following example instigated by Montesinos.

Example 18. The knots K_1 and K_2 of Fig. 39 have the same double branched cover, but distinct polynomials. The double branched cover of each of these knots is homeomorphic, preserving orientations induced by that of S^3, to the 3-manifold obtained by $+1$ surgery on the four-crossing knot [18]. Except for two of the uppermost crossings of K_1, K_1 and K_2 are mutually obverse. Focussing on one of those crossings, and using Proposition 10,

$$l^{-1}\mathscr{P}(K_1) + l\,\overline{\mathscr{P}(K_2)} = -m\mathscr{P}(L),$$

where L is the link obtained by nullifying that crossing. Now, it is easy to calculate that in $\mathscr{P}(L)$ the term in m^{-1} is

$$\mu l^8(2l^6 + 3l^4)$$

and as this is not self-conjugate with respect to interchanging l and l^{-1}, we cannot have $\mathscr{P}(K_1) = \mathscr{P}(K_2)$.

As mentioned in the introduction it is classical (see [1] pp. 277 and 302) that the Alexander polynomial $\Delta_K(t)$ is $\mathscr{P}(K)$ (i, $\mathrm{i}(t^{1/2} - t^{-1/2})$). Jones verifies that the polynomial $V_K(t)$ satisfies $V_{\mathscr{U}}(t) = 1$ and

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) + (t^{-1/2} - t^{1/2})V_{K_0}(t) = 0$$

where K_+, K_- and K_0 are a skein triple of closed braids [12]. Since any oriented link can be expressed as a closed braid, $V_K(t)$ can be calculated entirely within the closed braid context using the above formula and, of course, the calculation must produce

$$V_K(t) = \mathscr{P}(K)(\mathrm{i}t^{-1}, \mathrm{i}(t^{-1/2} - t^{1/2})).$$

One would expect that the two variable polynomial $\mathscr{P}(K)(l, m)$ would be a stronger invariant than the combination of its two specializations $\Delta_K(t)$ and $V_K(t)$. This is confirmed by the

Fig. 38.

Fig. 39.

following example which comes from the Thistlethwaite tabulations. (Those tabulations also show that there are several pairs of twelve-crossing non-alternating knots that can be distinguished by $\Delta_K(t)$ but not by $V_K(t)$.)

Example 19. Let K be the arborescent knot shown in Fig. 40, and let $\bar{K}$ be its obverse. Then $\mathscr{P}(K) \neq \mathscr{P}(\bar{K})$, but $V_K(t) = V_{\bar{K}}(t)$, $\Delta_K(t) = \Delta_{\bar{K}}(t)$.

Proof. $\mathscr{P}(K) = (3 + 5l^2 + 4l^4 + l^6) + (-4 - 10l^2 - 5l^4)m^2 + (1 + 6l^2 + l^4)m^4 - l^2m^6.$

This is (very) asymmetric in l and l^{-1}, so $\mathscr{P}(K) \neq \mathscr{P}(\bar{K})$. However substituting $l = \mathrm{i}t^{-1}$, $m = \mathrm{i}(t^{-1/2} - t^{1/2})$ gives

$$V_K(t) = t^{-2} - t^{-1} + 1 - t + t^2,$$

which is symmetric in t and t^{-1}. (The four crossing knot has the same Jones polynomial.) Thus $V_K(t) = V_{\bar{K}}(t)$. Further $\Delta_K(t) = \Delta_{\bar{K}}(t)$ whatever the knot K might be.

One final electronic calculation is as follows:

Example 20. The untwisted double of the left-hand trefoil with positive clasp, depicted in Fig. 41, has polynomial

$$(-l^{-2} + 4l^2 + 8l^4 + 5l^6 + l^8) + (1 - 5l^2 - 14l^4 - 10l^6 - l^8 + l^{10})m^2$$
$$+ (l^2 + 7l^4 + 6l^6)m^4 + (-l^4 - l^6)m^6.$$

Taking the untwisted double of a knot is a well known way of constructing a knot with $\Delta_K(t) = 1$. One way of proving that is by remarking that such a double has a Seifert matrix that is *also* a Seifert matrix for the unknot. The reason for quoting the above appalling polynomial is that it may serve as a cautionary tale in any quest for an understanding of the two-variable polynomial by means of Seifert matrices.

§3. ALGEBRAIC PROPERTIES OF THE NEW POLYNOMIAL

In this section we shall describe some of the elementary algebraic properties of the polynomial and further calculations some of which were inspired by Jones' calculations [12] for his one-variable polynomial.

For example we note that

$$\mathscr{P}(\mathscr{U}^c)(l, -(l + l^{-1})) = 1,$$

so that the recursive calculation of the polynomial from standard ascenders shows the following.

PROPOSITION 21. $\mathscr{P}(K)(l, -(l + l^{-1})) = 1.$

In the same spirit, but less easy, we have the following description of the basic terms of the polynomial. Let L denote an oriented link (or knot) of $c = c(L)$ components each of which,

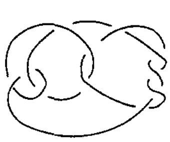


Fig. 40.

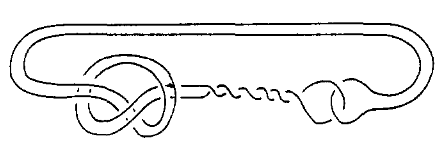


Fig. 41.

separately, is a knot $K_j, j = 1, \ldots, c$. Let $\lambda(K_j, K_k), j \neq k$, denote the linking number of the two components K_j and K_k. Define the *total linking* of L by

$$\lambda = \lambda(L) = \sum_{j<k} \lambda(K_j, K_k).$$

If we collect powers of m we may express the polynomials by

$$\mathscr{P}(L) = \sum p_k(l)m^k$$

and

$$\mathscr{P}(K_j) = \sum p_k^j(l)m^k.$$

PROPOSITION 22. *For an oriented link L, the powers of l and m which appear in $\mathscr{P}(L)$ are either all even or all odd, depending upon whether the number of components of K is odd or even, respectively. The exponent of the lowest power of m which appears is precisely $1-c$. It has coefficient*

$$p_{1-c}(l) = (-l^2)^{-\lambda}(-(l+l^{-1}))^{c-1} \prod_{j=1}^{c} p_0^j(l).$$

For a knot, $p_0(i) = 1$.

Proof. The proof is by induction on (n, s) where n is the number of crossings in a projection and s is the number of crossing switches necessary to achieve a standard ascending configuration (for some choice of basepoint and ordering of components). The pairs (n, s) are ordered lexicographically. Thus, given a projection of L with n crossings the result is assumed true for all projections of fewer crossings, and those of n crossings but smaller s.

We first note that the statements are true for any unlink because

$$\mathscr{P}(\mathscr{U}^c) = (-(l+l^{-1})m^{-1})^{c-1}$$
$$= (-(l+l^{-1}))^{c-1}m^{1-c}.$$

Suppose that proposition is true for the pair L_- and L_0, (the case L_+ and L_0 being analogous) and that we wish to verify the statements for L_+. Consider the fundamental identity

$$\mathscr{P}(L_+) = (-l^{-2})\mathscr{P}(L_-) - ml^{-1}\mathscr{P}(L_0).$$

L_0 has either one more or one less component than L_+ and L_-. It is convenient first to employ crossing changes between distinct components needed to achieve a separated union of the components (which may themselves still be knotted). For such a change the lowest power of m which occurs in L_0 is, by induction on n, $2 - c(L_+)$ as L_0 has one fewer component. Thus the contribution of the term $-ml^{-1}\mathscr{P}(L_0)$ will preserve the required parity of powers of l and m and will not contribute to the term in m^{1-c}. Since $\lambda(L_-) = \lambda(L_+) - 1$ we see, by induction on s, that the term $-l^{-2}\mathscr{P}(L_-)$ gives $p_{1-c}(l)$ as required.

In the remaining case when L_+ is a separated union of components we have, by Proposition 8, $\mathscr{P}(L_+) = \mu^{c-1} \prod_{j=1}^{c} \mathscr{P}(K_j)$. Thus, if the proposition is true for single components, the lowest power of m is precisely $1-c$, the parity of the powers of l and m satisfy the requirements, and the coefficient of m^{1-c} is given by the formula (since $\lambda = 0$).

Hence, consider the case when L_+ is a single component. We observe that, $L_0 = \{K_{0,1}, K_{0,2}\}$, is a link of two components whose lowest term is, by induction on n, $(-l^2)^{-\lambda(L_0)}(-(l+l^{-1}))p_0^{0,1}(l)p_0^{0,2}(l)m^{-1}$. Thus, the zeroth power of m in $\mathscr{P}(L_+)$ is the sum of $-l^{-1}(-l^2)^{-\lambda(L_0)}(-(l+l^{-1}))p_0^{0,1}(l)p_0^{0,2}(l)$ with the zeroth term of $-l^{-2}(\mathscr{P}(L_-))$. Since the former term gives zero when $l = i$ we have, by induction on s, that $p_0(i) = 1$ as required. Thus we have the correct lowest order term and, as above, the correct parity for the powers of l and m for a knot. This, in turn, provides the correct lowest term for a separated link as we have shown that the lowest nonzero term for the polynomial associated to each component is precisely the 0th power of m.

Remark. That $p_0(i) = 1$ for a knot is just the statement of the familiar $\Delta_K(1) = 1$.

We employ this proposition to show the existence and isotopy invariance of the *total twisting*, $\tau(K)$, of a knot K defined as follows. Given a sequence of crossing changes (of a projection of K) $\sigma_j, j = 1, \ldots, n$, from crossings of sign $\varepsilon_j, j = 1, \ldots, n$, which unknot K (such as in the definition of $\mathscr{P}(K)$) there is an associated sequence of two component links

$$L_j = \left(\eta_j \prod_{k<j} \sigma_k\right)(K) = \{L_j^0, L_j^1\}, j = 1, \ldots, n,$$

from which we may define

$$\tau(K) = \sum \varepsilon_j \lambda(L_j^0, L_j^1).$$

Recall that η_j is the operation of nullifying the jth crossing. We show that $\tau(K)$ can be calculated from $\mathscr{P}(K)$ and is therefore well defined and an isotopy invariant of K.

Proposition 23. Suppose that K is a knot and $\mathscr{P}(K)(l, m) = \sum p_j(l)m^j$, then

(i) $p_0'(i) = 0$

(ii) $p_0''(i) = 8\tau(K)$.

Proof. (i) Suppose that K has a projection with n crossings that require s crossing switches to achieve a standard ascending projection. Again we induct on (n, s). Suppose switching the first of these crossings, of sign ε, changes K to $\hat{K}$, and nullifying it gives the link L with components K_1 and K_2. Then $\mathscr{P}(K) = -l^{2\varepsilon}\mathscr{P}(\hat{K}) - ml^{-\varepsilon}\mathscr{P}(L)$. Thus, using Proposition 22,

$$p_0(l) = -l^{-2\varepsilon}\hat{p}_0(l) + l^{-\varepsilon}(-l^2)^{-\lambda}(l + l^{-1})p_0^1(l)p_0^2(l).$$

By induction the result is true for $\hat{p}_0$, p_0^1 and p_0^2, so, differentiating with respect to l and substituting $l = i$ gives

$$p_0'(i) = 2\varepsilon(i)^{-2\varepsilon-1}\hat{p}_0(i) + i^{-\varepsilon}(1 - i^{-2})p_0^1(i)p_0^2(i).$$

(ii) The method of evaluating $\tau(K)$ described above depends on a sequence of r crossing switches that change K to the unknot. Suppose inductively the result is true for all calculations of τ by all sequences of $(r-1)$ crossing switches that reduce knots to unknots. As above suppose the first switch of such a sequence of r switches changes K to $\hat{K}$, that the relevant crossing has sign ε, and that nullifying the crossing gives the link L. As before

$$p_0(l) = -l^{-2\varepsilon}\hat{p}_0(l) + l^{-\varepsilon}(-l^2)^{-\lambda}(l + l^{-1})p_0^1(l)p_0^2(l).$$

Differentiating twice, substituting $l = i$, using the last clause of Proposition 22, and using (i) above, one obtains $p_0''(i) = \hat{p}_0''(i) + 8\varepsilon\lambda$. However, by the induction $\hat{p}_0''(i) = 8\tau(\hat{K})$, and so $p_0''(i) = 8\tau(K)$ where $\tau(K)$ is calculated via the given sequence of r switches. This completes the induction and the proof.

COROLLARY. *$\tau(K)$ is well defined, i.e. is independent of the calculation sequence.*

We note that if K is a τ-twisted double of a knot, then $\tau(K) = \tau$.

PROPOSITION 24. (i) *If L is a link of $c \geqq 2$ components then*

$$[(-(l+l^{-1}))^{2-c}p_{3-c}(l)]_{l=i} = \lambda(L)i.$$

(ii) *If K is a knot then*

$$p_2(i) = -\tau(K).$$

Proof. (i) The proof is by induction on the number of crossing changes required to change L to a separated union of, possibly knotted, components. The case of no changes follows from the fact that, in that case,

$$\mathscr{P}(L) = (-(l+l^{-1}))^{c-1}m^{1-c}\prod_j \mathscr{P}(K_j)$$

so that

$$[(-(l+l^{-1}))^{2-c}p_{3-c}(l)] = (-(l+l^{-1}))\left(\sum_j p_2^j(l)\prod_{k\neq j} p_0^k(l)\right),$$

where $p_i^j(l)$ denotes the coefficient of m^i in the polynomial for K_j. By virtue of the first term, evaluation at i gives zero.

In general we compute (from Proposition 22) that

$$p_{3-c}(l) = -l^{-2\varepsilon}\hat{p}_{3-c}(l) - l^{-\varepsilon}(-l^2)^{-\lambda(\tilde{L})}(-(l+l^{-1}))^{c-2}\prod_j \tilde{p}_0^j(l)$$

where $\hat{L}$ is the link formed by switching one of the crossings of sign ε between distinct components, $\tilde{L}$ the result of nullifying it and $\{\tilde{K}_j\}$ denotes the components of $\tilde{L}$. Hence

$$[-(l+l^{-1})^{2-c}p_{3-c}(l)] = [-l^{-2\varepsilon}(-(l+l^{-1}))^{2-c}\hat{p}_{3-c}(l)] - l^{-\varepsilon}(-l^2)^{-\lambda(\tilde{L})}\prod_j \tilde{p}_0^j(l),$$

which, upon evaluation (using the induction) at $l = i$, gives

$$\lambda(\hat{L})i + \varepsilon i = \lambda(L)i.$$

(ii) If K is a knot we calculate inductively on the number of crossing changes needed to achieve the unknot and employ the previous formula for the two component link, L, that results from nullifying a crossing of sign ε as follows:

$$p_2(l) = -l^{2\varepsilon}\hat{p}_2(l) - l^{-\varepsilon}p_1^L(l)$$

so that

$$p_2(i) = \hat{p}_2(i) - \varepsilon\lambda(L)$$
$$= -\tau(K).$$

Remark. For a knot we note that $-p_2(i)$ is the second coefficient in the Conway potential function. The reduction modulo 2 of $-p_2(i)$ gives the Arf or Kervaire invariant. A proof appears in [13].

§4. QUESTIONS AND TABLES

The axiomatic description of the polynomial given in the theorem of this paper is natural enough if one concentrates on the idea of changing crossings in link projections, however the proof of existence of the polynomial consists entirely of combinatorics. It may be that the polynomial, though an isotopy invariant, must rest on combinatorics and that there is no other truth at its foundations. That however seems contrary to the spirit and traditions of algebraic topology, hence the first question:

Question 1. Can the two-variable polynomial be defined in terms of fundamental groups, homology groups and covering spaces? For a knot, how is it related to the knot group and its peripheral subgroup? If, as seems unlikely, there be two distinct knots (not mutually reverse) with homeomorphic oriented complements do they have the same two-variable polynomials?

Question 2. Can the polynomial be defined for links in a homology 3-sphere?

The algebraic form of the knot discussed in §3 gives very little information concerning the general style of the polynomial, and the tables of evaluations for low crossing number simply suggest that the polynomial becomes more unpleasant as the number of crossings increases.

Question 3. Is there a simple precise algebraic characterization of the elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ that occur as polynomials of oriented links?

Question 4. Is there a non-trivial knot K for which $\mathscr{P}(K) = 1$? Is there a non-trivial link L of c components for which $\mathscr{P}(L) = \mu^{c-1}$?

The Alexander polynomial can be defined fairly easily for knots and links of high dimension, and it gives information about cobordism.

Question 5. Is there an analogous definition of a 2-variable polynomial for links of 2-spheres in the 4-sphere?

Question 6. Does the two-variable polynomial give any new information about classical link cobordism? Is there a two-variable analogue of the condition of Fox and Milnor that a slice knot has Alexander polynomial of the form $f(t)f(t^{-1})$?

Of course, the Alexander polynomial vanishes for a boundary link, namely a link of two or more components whose components bound *disjoint* Seifert surfaces (see, for example [19]).

Question 7. Does the two-variable polynomial of a boundary link have any special form?

As the definition of the polynomial concerns crossing switches the next question *seems* natural.

Question 8. Does the new polynomial give new information about the unknotting number of any knots (e.g. torus knots)?

When some of these questions have been resolved it may be fruitful to consider the question of what is the best notation for the two-variable polynomial.

The nature of the new polynomial renews interest in the idea of skein-equivalence. The following questions are thought to be unanswered.

Question 9. Is there a non-trivial knot or link that is skein equivalent to the trivial knot or link?

Question 10. Can two knots with distinct unknotting numbers be skein-equivalent? Can one be a mutant of the other?

It seems desirable to record a tabulation of the two-variable polynomial for knots and links of low crossing number, if only to gain a little experimental insight into the form of the polynomial and to provide a basis for individual calculations for links of some specific interest. Much of the information in the tables given below has been obtained with, or confirmed by, computers (including that of Thistlethwaite who will publish much more grandiose tabulations on microfiche [21]). Interpretation of the tables is as follows: Knots are listed with the classical Alexander–Briggs notation $3_1, 4_1, 5_1, 5_2, \ldots, 9_{48}, 9_{49}$, and a "coded" form of the polynomial is given. The polynomial of a *knot* is of the form $\sum_{i\geq 0} p_i(l)m^i$ where $p_i(l) = 0$ if i is odd. The numbers in the ith rounded bracket give the coefficients in $p_{2(i-1)}(l)$, the number in square brackets being the coefficient of l^0, and as $p_i(l)$ contains only even powers of l, no entry occurs for the coefficient of an odd power.

Example

$$\mathscr{P}(9_{17}) = (-2[-3]-2)(1[6]5\ 2)([-2]-4\ -1)([0]1)$$
$$= (-2l^{-2}-3-2l^2)+(l^{-2}+6+5l^2+2l^4)m^2+(-2-4l^2-l^4)m^4+l^2m^6.$$

For a knot (of *one* component) the polynomial is unchanged by reversing the orientation so there is no need to specify a direction on the knot. One does however need to distinguish between a knot and its obverse, the conventions used here are those of the pictures of knots given at the end of Rolfsen's book [18]. Recall that the obverse of a knot has polynomial conjugate to that of the knot. For example,

$$\mathscr{P}(\overline{9_{17}}) = (-2[-3]-2)(2\ 5\ [6]\ 1)(-1\ -4\ [-2])(1\ [0]).$$

3_1 ([0] −2 −1) ([0] 1)
4_1 (−1 [−1] −1) ([1])
5_1 ([0] 0 3 2) ([0] 0 −4 −1) ([0] 0 1)
5_2 ([0] −1 1 1) ([0] 1 −1)
6_1 (−1 [0] 1 1) ([1] −1)
6_2 ([2] 2 1) ([−1] −3 −1) ([0] 1)
6_3 (1 [3] 1) (−1 [−3] −1) ([1])

7_1 ([0] 0 0 −4 −3) ([0] 0 0 10 4) ([0] 0 0 −6 −1) ([0] 0 0 1)
7_2 ([0] −1 0 −1 −1) ([0] 1 −1 1)
7_3 (−2 −2 1 0 [0]) (1 3 −3 0 [0]) (−1 1 0 [0])
7_4 (−1 0 2 0 [0]) (1 −2 1 [0])
7_5 ([0] 0 2 0 −1) ([0] 0 −3 2 1) ([0] 0 1 −1)
7_6 ([1] 1 2 1) ([−1] −2 −2) ([0] 1)
7_7 (1 2 [2]) (−2 [−2] −1) ([1])

8_1 (−1 [0] 0 −1 −1) ([1] −1 1)
8_2 ([0] −3 −3 −1) ([0] 4 7 3) ([0] −1 −5 −1) ([0] 0 1)
8_3 (1 0 [−1] 0 1) (−1 [2] −1)
8_4 (−2 [−2] 0 1) (1 [3] −2 −1) ([−1] 1)
8_5 (−2 −5 −4 [0]) (3 8 4 [0]) (−1 −5 −1 [0]) (1 0 [0])
8_6 ([2] 1 −1 −1) ([−1] −2 2 1) ([0] 1 −1)
8_7 (−2 −4 [−1]) (3 8 [3]) (−1 −5 [−1]) (1 [0])

8_8 (−1 −1 [2] 1) (1 2 [−2] −1) (−1 [1])
8_9 (−2 [−3] −2) (3 [8] 3) (−1 [−5] −1) ([1])
8_{10} (−3 −6 [−2]) (3 9 [3]) (−1 −5 [−1]) (1 [0])
8_{11} ([1] −1 −2 −1) ([−1] −1 2 1) ([0] 1 −1)
8_{12} (1 1 [1] 1 1) (−2 [−1] −2) ([1])
8_{13} ([0] −2 −1) (−1 [−1] 2 1) ([1]−1)
8_{14} ([1]) ([−1] −1 1 1) ([0] 1 −1)

8_{15} ([0] 0 1 −3 −4 −1) ([0] 0 −2 5 3) ([0] 0 1 −2)
8_{16} ([0] −2 −1) ([2] 5 2) ([−1] −4 −1) ([0] 1)
8_{17} (−1 [−1] −1) (2 [5] 2) (−1 [−4] −1) ([1])
8_{18} (1 [3] 1) (1 [1] 1) (−1 [−3] −1) ([1])
8_{19} (−1 −5 −5 0 0 [0]) (5 10 0 0 [0]) (−1 −6 0 0 [0]) (1 0 0 [0])
8_{20} ([−1] −4 −2) ([1] 4 1) ([0] −1)
8_{21} ([0] −3 −3 −1) ([0] 2 3 1) ([0] 0 −1)

9_1 ([0] 0 0 0 5 4) ([0] 0 0 0 −20 −10) ([0] 0 0 0 21 6) ([0] 0 0 0 −8 −1) ([0] 0 0 0 1)
9_2 ([0] −1 0 0 1 1) ([0] 1 −1 1 −1)
9_3 (3 3 −1 0 0 [0]) (−4 −7 6 0 0 [0]) (1 5 −5 0 0 [0]) (−1 1 0 0 [0])
9_4 ([0] 0 1 0 2 2) ([0] 0 −3 2 −3 −1) ([0] 0 1 −1 1)
9_5 (1 0 −1 1 0 [0]) (−1 2 −2 1 [0])
9_6 ([0] 0 0 −3 −1 1) ([0] 0 0 7 −3 −3) ([0] 0 0 −5 4 1) ([0] 0 0 1 −1)
9_7 ([0] 0 2 1 1 1) ([0] 0 −3 1 −2 −1) ([0] 0 1 −1 1)

9_8 (−1 [−1] 0 2 1) (1 [2] −1 −2) ([−1] 1)
9_9 ([0] 0 0 −2 1 2) ([0] 0 0 7 −4 −3) ([0] 0 0 −5 4 1) ([0] 0 0 1 −1)
9_{10} (2 1 −2 0 0 [0]) (−1 −2 5 −2 0 [0]) (1 −2 1 0 [0])
9_{11} (−2 −3 −1 −1 [0]) (1 6 4 3 [0]) (−2 −4 −1 [0]) (1 0 [0])
9_{12} ([1] 0 −1 −2 −1) ([−1] −1 1 2) ([0] 1 −1)
9_{13} (1 −1 −3 0 0 [0]) (−1 −1 5 −2 0 [0]) (1 −2 1 0 [0])
9_{14} (−1 −2 −1 [1]) (2 1 [−1] −1) (−1 [1])

9_{15} (−1 −1 1 1 [1]) (2 0 −1 [−1]) (−1 1 [0])
9_{16} (−3 −4 0 0 [0]) (−2 0 8 0 0 [0]) (1 3 −5 0 0 [0]) (−1 1 0 0 [0])
9_{17} (−2 [−3] −2) (1 [6] 5 2) ([−2] −4 −1) ([0] 1)
9_{18} ([0] 0 1 −1 0 1) ([0] 0 −2 4 −1 −1) ([0] 0 1 −2 1)
9_{19} (1 1 [0] −1) (−2 [0] 1 1) ([1] −1)
9_{20} ([0] −2 −2 −2 −1) ([0] 3 5 5 1) ([0] −1 −4 −2) ([0] 0 1)
9_{21} (−1 −1 0 −1 [0]) (2 0 0 [−1]) (−1 1 [0])

9_{22} (−1 −4 [−4] −2) (2 6 [6] 1) (−1 −4 [−2]) (1 [0])
9_{23} ([0] 0 1 −2 −2) ([0] 0 −2 4 0 −1) ([0] 0 1 −2 1)
9_{24} (−1 [−3] −5 −2) (2 [6] 6 1) (−1 [−4] −2) ([1])
9_{25} ([1] −1 −3 −3 −1) ([−1] 0 4 3) ([0] 1 −2)
9_{26} (−1 −3 −3 [0]) (1 5 6 [2]) (−2 −4 [−1]) (1 [0])
9_{27} (−1 [−2] −3 −1) (2 [6] 5 1) (−1 [−4] −2) ([1])
9_{28} ([−1] −5 −4 −1) ([2] 7 5 1) ([−1] −4 −2) ([0] 1)

9_{29} (−1 [−3] −5 −2) (1 [5] 7 2) ([−2] −4 −1) ([0] 1)
9_{30} (−2 [−4] −4 −1) (2 [7] 5 1) (−1 [−4] −2) ([1])
9_{31} ([−1] −4 −2) ([2] 7 4 1) ([−1] −4 −2) ([0] 1)

9_{32} (−1 −2 −1 [1]) (1 4 3 [1]) (−2 −3 [−1]) (1 [0])
9_{33} ([0] −2 −1) (1 [3] 4 1) (−1 [−3] −2) ([1])
9_{34} (−1 [−1] −1) (1 [4] 3 1) (−1 [−3] −2) ([1])
9_{35} ([0] 0 0 −3 −1 1) ([0] 1 −2 3 −1)

9_{36} (−2 −4 −3 −2 [0]) (1 6 5 3 [0]) (−2 −4 −1 [0]) (1 0 [0])
9_{37} (1 0 [−2] −2) (−2 [1] 1 1) ([1] −1)
9_{38} ([0] 0 0 −4 −3) ([0] 0 −1 7 1 −1) ([0] 0 1 −3 1)
9_{39} (−1 −2 −2 −2 [0]) (3 3 1 [−1]) (−2 1 [0])
9_{40} ([2] 2 1) ([0] 0 2 1) ([−1] −2 −2) ([0] 1)
9_{41} ([0] −3 −3 −1) (−1 [0] 4 3) ([1] −2)
9_{42} (−2 [−3] −2) (1 [4] 1) ([−1])

9_{43} (−1 −3 −4 −3 [0]) (4 7 4 [0]) (−1 −5 −1 [0]) (1 0 [0])
9_{44} (−1 [−2] −3 −1) ([2] 3 1) ([0] −1)
9_{45} ([0] −2 −2 −2 −1) ([0] 2 2 2) ([0] 0 −1)
9_{46} ([2] 1 −1 −1) ([0] −1 1)
9_{47} (−1 −2 −1 [1]) (3 4 [2]) (−1 −4 [−1]) (1 [0])
9_{48} (2 3 0 [0]) (−3 −1 [−1]) (1 [0])
9_{49} (−3 −4 0 0 [0]) (2 6 −2 0 [0]) (−2 1 0 [0]).

The question of orientations of links is more confusing. There does not seem to be a set of diagrams of links in the literature with orientations marked. In what follows the tables refer to Rolfsen's diagrams in [18], and to his n^i_j nomenclature; Conway's notation is also recorded. In one of Rolfsen's diagrams of two-component links there are four ways of inserting arrows on the components. However, changing all arrows leaves the polynomial unchanged, so there are at most two polynomials corresponding to each diagram. They are both given in the tables if they are distinct, otherwise the unique polynomial is recorded. Recall that, for a two-component oriented link, the coefficient of m^{-1} is $-(-l^2)^{-\lambda}(l^{-1}+l)p^1_0(l)p^2_0(l)$ where λ is the linking number of the two components, $p^1_0(l)$ and $p^2_0(l)$ being the first terms (coefficients of m^0) in the polynomials of those components. In most of the early examples these components are unknotted so that $p^1_0(l) = 1 = p^2_0(l)$. Then, for a given choice of arrows, the first term is $-(-l^2)^{-\lambda}(l^{-1}+l)m^{-1}$. Since λ can be found by adding the signs of the cross-overs where component 1 passes under component 2, it is easy to determine which of the two choices is the appropriate polynomial. If $\lambda = 0$ that method fails but in the table given below there is then only one choice of polynomial anyway! This leaves links 7^2_5 and 7^2_7. Here similar reasoning works, the first terms being $(-l^2)^{-\lambda}(2l+3l^3+l^5)m^{-1}$ and $(-l^2)^{-\lambda}(2l^{-1}+3l^{-3}+l^{-5})m^{-1}$ respectively.

For a link of two components, only odd powers of l and m appear in the polynomial, and the first term is in m^{-1}. In the coding given for these polynomials, the numbers in a pair of round brackets are the coefficients of powers of l that form the polynomial coefficient of a power of m. The asterisks separate negative and positive powers of l.

Example

(1 1 *)(−1 −2 * 1 1)(1 * −1) = $(l^{-3}+l^{-1})m^{-1}+(-l^{-3}-2l^{-1}+l+l^3)m+(l^{-1}-l)m^3$.

2^2_1 2 (* 1 1) (* −1)
(1 1 *) (−1 *)

4^2_1 4 (* 0 −1 −1) (* 0 3 1) (* 0 −1)
(−1 −1 0 *) (1 −1 *)

5^2_1	212	(−1 * −1) (1 * 2 1) (* −1)
6^2_1	6	(* 0 0 1 1) (* 0 0 −6 −3) (* 0 0 5 1) (* 0 0 −1) (1 1 0 0 *) (−1 1 −1 *)
6^2_2	33	(1 1 0 0 *) (−1 −2 2 0 *) (1 −1 0 *) (* 0 0 1 1) (* 0 2 −2 −1) (* 0 −1 1)
6^2_3	222	(* 0 −1 −1) (* 0 2 −1 −1) (* 0 −1 1) (−1 −1 0 *) (2 1 * 1) (−1 *)
7^2_1	412	(* 1 1) (* −3 −4 −2) (* 1 4 1) (* 0 −1) (1 1 *) (−1 −2 * 1 1) (1 * −1)
7^2_2	3112	(1 1 *) (−2 −5 * −2) (1 4 * 1) (−1 *) (* 1 1) (1 * 0 −1 −1) (* −1 1)
7^2_3	232	(−1 * −1) (1 * 1 −1 −1) (* −1 1)
7^2_4	3, 2, 2	(1 3 2 *) (−2 −5 −3 *) (1 4 1 *) (−1 0 *)
7^2_5	21, 2, 2	(* 0 0 2 3 1) (* 0 1 −4 −3) (* 0 −1 2) (2 3 * 1) (−2 −6 * −2) (1 4 * 1) (−1 *)
7^2_6	.2	(−1 * −1) (−1 * −2 −1) (1 * 3 1) (* −1)
7^2_7	3, 2, 2−	(1 3 2 0 0 *) (−4 −6 0 0 *) (1 5 0 0 *) (−1 0 0 *) (1 * 3 2) (−1 * −4 −1) (* 1)
7^2_8	21, 2, 2−	(* 2 3 1) (* −2 −3 −1) (* 0 1).

The situation for three component links is analogous to the previous cases in that the powers of l and m are even and the lowest power of m is m^{-2} having coefficient $-((-l)^2)^{-\lambda}(l^{-1}+l)^2 p_0^1(l)p_0^2(l)p_0^3(l)$ where λ is the total linking number of §3. In the lowest crossing number cases the various components are all unknotted so that $p_0^1(l)=p_0^2(l)=p_0^3(l)=1$. In this range, there are at most two polynomials associated with the various choices of orientations. The linking number will determine which is appropriate except in the case where it is zero. In this case, however, there is only one choice of polynomial. The table employs the notation for knots since all powers are even, except that the first term determines the coefficient of m^{-2}.

Example

(−1 −2 [−1]) (1 3 [2]) (−1 −3 [−1]) (1 [0]) $= (-l^{-4}-2l^{-2}-1)m^{-2}+(l^{-4}+3l^{-2}+2) + (-l^{-4}-3l^{-2}-1)m^2+l^{-2}m^4$

6^3_1	2, 2, 2	(−1 −2 [−1]) (1 3 [2]) (−1 −3 [−1]) (1 [0]) ([0] 0 −1 −2 −1) ([0] 0 3 3) ([0] 1 −2)
6^3_2	(.1	(1 [2] 1) ([0]) (−1 [−2] −1) ([1]))
6^3_3	2, 2, 2−	([−1] −2 −1) ([2] 3 1) ([0] −1) (−1 −2 −1 0 [0]) (3 3 0 [0]) (−1 −4 0 [0]) (1 0 [0])
7^3_1	2, 2, 2+	([−1] −2 −1) ([2] 3 1) (−1 [−2] −2) ([1]) (−1 −2 −1 0 [0]) (3 3 0 [0]) (1 1 −3 0 [0]) (−1 1 0 [0]).

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