PROBLEM 1 (20 Points) Prove the following:

1. (10 Points) For any vector $x \in \mathbb{R}^n$

$$\frac{\|x\|_1}{n} \leq \|x\|_\infty \leq \|x\|_1$$

Solution:

The first inequality

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} (\max_i |x_i|) = \sum_{i=1}^{n} \|x\|_\infty \leq n\|x\|_\infty$$

and the second

$$\|x\|_\infty = \max_i |x_i| \leq \sum_{i=1}^{n} |x_i| = \|x\|_1$$

2. (10 Points) For any matrix $A \in \mathbb{R}^{n\times n}$

$$\frac{\|A\|_1}{n} \leq \|A\|_\infty \leq n\|A\|_1$$

Solution:

We know that $\|A\|_1$ is the 1-norm of the column with the biggest 1-norm, let us suppose that this column is the p-th column, that is,

$$\|A\|_1 = \max_{j=1:n} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ip}|.$$

Now, the absolute value of each element is less or equal than the sum of the absolute values of all the elements in that row, that is the 1-norm of that row

$$|a_{ip}| \leq \sum_{j=1}^{n} |a_{ij}|.$$

Therefore

$$\|A\|_1 = \sum_{i=1}^{n} |a_{ip}| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{n} \max_{i=1:n}(\sum_{j=1}^{n} |a_{ij}|) = \sum_{i=1}^{n} \|A\|_\infty = n\|A\|_\infty$$

The first inequality

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} (\max_i |x_i|) = \sum_{i=1}^{n} \|x\|_\infty \leq n\|x\|_\infty$$

The reasoning to prove $\|A\|_\infty \leq n\|A\|_1$ is exactly the same. It can be done also using the first inequality and that $\|A\|_\infty = \|A^T\|_1$. The first inequality for A^T is

$$\frac{\|A^T\|_1}{n} \leq \|A^T\|_\infty,$$

that is

$$\frac{\|A\|_\infty}{n} \leq \|A\|_1.$$

But maybe a more elegant proof is using the inequalities for vectors above and the definition of the induced norms:

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \max_{x\neq 0} \frac{n\|Ax\|_\infty}{\|x\|_\infty} = n\|A\|_\infty$$

and

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{x\neq 0} \frac{\|Ax\|_1}{\frac{\|x\|_1}{n}} = n\|A\|_1$$

where the inequalities in both equations come from using the inequalities in the vector norms, one for the denominator and the other for the numerator.

PROBLEM 2 (20 Points)

1. (10 Points) Compute, if possible, the LDL^T decomposition, (L unit lower triangular and D diagonal and invertible) with no pivoting, of the symmetric matrix

$$A = \begin{bmatrix} 2 & -4 & 4 & 0 \\ -4 & 11 & -8 & -6 \\ 4 & -8 & 9 & -2 \\ 0 & -6 & -2 & 18 \end{bmatrix}$$

Solution:

The LU factorization of A gives

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & -4 & 4 & 0 \\ 0 & 3 & 0 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Therefore $A = LDL^T$ with

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

2. (10 Points) Is A positive definite? Why? If the answer was "yes", compute the Cholesky factor of A, that is, a lower triangular matrix R such that $A = RR^T$

Solution:

A is positive definite because we can do the LDL^T decomposition with all the elements of D positive (that is all the pivots in the LU decomposition are positive, or the principal minors of A are positive).

The Cholesky factor is just $R = L\sqrt{D}$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ -2\sqrt{2} & \sqrt{3} & 0 & 0 \\ 2\sqrt{2} & 0 & 1 & 0 \\ 0 & -2\sqrt{3} & -2 & \sqrt{2} \end{bmatrix}$$

PROBLEM 3 (20 Points)

1. (10 Points) Find $\alpha > 0$ and $\beta > 0$ such that the matrix

$$A = \begin{bmatrix} 3 & 2 & \beta \\ \alpha & 5 & \beta \\ 2 & 1 & \alpha \end{bmatrix}$$

is strictly diagonally dominant.

Solution:

The conditions on the three rows of A for it to be strictly diagonally dominant are

$$\begin{aligned} 3 &> 2 + \beta \\ 5 &> \alpha + \beta \\ \alpha &> 3 \end{aligned}$$

Therefore the conditions are

$$\begin{aligned} &0 < \beta < 1 \\ &3 < \alpha < 5 - \beta \end{aligned}$$

2. (10 Points) Suppose A and B are symmetric positive definite matrices.

   1. Is $A + B$ positive definite?
      Solution:
      Yes, it is because $x^T(A + B)x = x^TAx + x^TBx > 0$ for any nonzero vector x.
   2. Is $A - B$ positive definite?
      Solution:
      No, it isn't. Take for example any A symmetric positive definite and $B = A$.