INTRODUCTION TO NUMERICAL ANALYSIS - MATH 104B SUMMER 2011

FINAL EXAM 9/7/2011 DATE DUE: WEDNESDAY, SEPTEMBER 7TH, AT 12:30 AM

NAME:

PROBLEM 1	PROBLEM 2	PROBLEM 3	PROBLEM 4	GRADE

PROBLEM 1 (40 Points) Let $v_1 = [1, 2, 2]^T$ y $v_2 = [5, 2, 0]^T$ be two vectors in $\mathbb{R}^3$.

1. **(10 Points)** Build an orthonormal basis of the subspace $Span[v_1, v_2]$.
2. **(10 Points)** Given the matrix $A = [v_1\, v_2] \in \mathbb{R}^{(3,2)}$, whose columns are the vectors v_1 and v_2, get its QR decompositon.
3. **(20 Points)** Suppose that a physical quantity E depends on the variables X and Y according to the law $E = aX + bY$, get the coefficients a y b using a least squares fit of the following data:

X	Y	E
1	5	1
2	2	1
2	0	1

PROBLEM 2 (40 Points) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}$$

Find all values of α such that the matrix A:

1. **(10 Points)** Is singular.
2. **(10 Points)** Is strictly diagonally dominant.
3. **(10 Points)** Is symmetric.
4. **(10 Points)** Is positive definite.

PROBLEM 3 (40 Points) Let

$$G = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

where $c = \cos\theta$ and $s = \sin\theta$, with $\theta \in \mathbb{R}$. (G is called a Givens matrix or rotation.)

1. **(10 Points)** Prove that G is orthogonal (it has orthonormal columns). If $z = Gx$, which is the relation between $\|x\|_2$ and $\|z\|_2$?

2. **(20 Points)** Prove that for any nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, there exists a Givens matrix $G \in \mathbb{R}^{2\times 2}$ such that $Gx = \begin{bmatrix} y \\ 0 \end{bmatrix}$ with $y > 0$. That is, compute the neccesary values of $c = \cos\theta$, $s = \sin\theta$ (you dont need to get the value of the angle θ) and y, as a function of the components of the vector x. For simplicity suppose that both x_1 and x_2 are different from zero.

3. **(10 Points)** Suppose that $A \in \mathbb{R}^{2\times n}$ and $G \in \mathbb{R}^{2\times 2}$ is an arbitrary Givens rotation, how many floating operations (additions+substractions+multiplications+divisions, count all the same) do you have to do to multiply GA?

PROBLEM 4 (40 Points) To solve the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is an invertible matrix with elements a_{ij}, the following iterative method (Richardson's) is proposed:

$$x_i^{(k+1)} = -a_{i,1}x_1^{(k)} - \cdots - a_{i,i-1}x_{i-1}^{(k)} + (1 - a_{i,i})x_i^{(k)} - a_{i,i+1}x_{i+1}^{(k)} - \cdots - a_{i,n}x_n^{(k)} + b_i \quad 1 \le i \le n$$

1. **(10 Points)** Write the iteration as

$$x^{(k+1)} = T_R x^{(k)} + c,$$

that is, give T_R and c in terms of A and b.

2. **(10 Points)** Suppose that A is symmetric and positive definite, then it is known that its eigenvalues are real and positive[1], prove that the spectral radius of the iteration matrix, T_R, of the Richardson's method is

$$\rho(T_R) = \max\{|1 - \lambda_1|, |1 - \lambda_n|\},$$

where λ_1 and λ_n are, respectively, the biggest and the smallest eigenvalues of A.

3. **(10 Points)** Show that, when A is symmetric positive definite, the Richardson's method converges if and only if $\lambda_1 < 2$.

4. **(10 Points)** If a symmetric positive definite matrix has $\lambda_1 = 1.1$ and $\lambda_n = 0.9$, how many iterations, k, of Richardson's method are neccesary to reduce the error in a factor 10^{-3}, that is,

$$\frac{\|x^{(k)} - x\|}{\|x^{(0)} - x\|} < 10^{-3},$$

being x the exact solution of the system.

[1] That is, if $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n > 0$