# Homework 3 – Math 104B, Summer 2011

**Due on Monday, August 29th, 2011**

**Section 7.2:** 4.c, 4.f, 8.c, 8.f, 9 and 13.

**Section 7.4:** 2.a, 4.a and 5.a.

**Programming Problem:** Consider the tridiagonal matrix $\mathbf{A} = (a_{i,j})_{1\leq i,j\leq n}$ given by

$$a_{i,j} = \begin{cases} -\frac{1}{h^2} & |i-j| = 1 \\ \frac{2}{h^2} & i = j \\ 0 & Otherwise \end{cases} \tag{1}$$

obtained when the following ODE,

$$\begin{aligned} -u''(x) &= f(x), \quad x \in [0,1] \\ u(0) &= u(1) = 0, \end{aligned} \tag{2}$$

is discretized using second order centered differences:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i) \quad i = 1, 2, \ldots, n, \tag{3}$$

where $h = 1/(n+1)$.

1. For each $k = 1, 2, \ldots, n$, show that the vector $\mathbf{u}^{(k)}$ given by

$$u_i^{(k)} = \sin\left(\frac{\pi k i}{n+1}\right), \quad i = 1, 2, \ldots, n \tag{4}$$

   is an eigenvector of the matrix $A$, and determine the corresponding eigenvalue $\lambda_k$.

2. Set up the Jacobi iteration for system (**??**), and show that the vectors (**??**) are also eigenvectors of the Jacobi iteration matrix, $\mathbf{T_J}$.

3. Determine the spectral radius of $\mathbf{T_J}$, $\rho(\mathbf{T_J})$.

4. The Jacobi iteration can be written as

$$\mathbf{x}^{(k+1)} = \mathbf{T_J}\mathbf{x}^{(k)} + \mathbf{c} \tag{5}$$

From the previous steps, we know that $\mathbf{T_J}$ is symmetric and diagonalizable. For a symmetric matrix $A$ it can be shown that

$$\|A\|_2 = \rho(A). \tag{6}$$

Therefore, as it was shown in class, we can write

$$\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_2 \leq \rho(\mathbf{T_J})^k \|\mathbf{x}^{(0)} - \mathbf{x}^*\|_2 \tag{7}$$

where $\mathbf{x}^*$ is the (unique) fixed point of (**??**).

Use formula (**??**) and the spectral radius obtained earlier to estimate the number of iterations necessary for the error to be less than a given $\epsilon$ as a function of the number of grid points used, $n$. You should end up with a formula of the form $number\_of\_iter = O(n^\alpha)$ for some $\alpha$.

5. Fix `tol` $= 10^{-4}$. Consider the vector $\mathbf{u}$ such that

$$u_i = \sin\left(\frac{\pi i}{n+1}\right) \tag{8}$$

and construct the right hand side $\mathbf{f} = \mathbf{A}\mathbf{u}$. Solve the system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{f} \tag{9}$$

using Jacobi's method with $\mathbf{x}^{(0)} = 0$. Use the values $n = 10, 20, 40, 80, 160, 320$. Do a log-log plot of the number of Jacobi iterations necessary for the error to satisfy

$$\|\mathbf{x}^{(k)} - \mathbf{u}\|_2 \leq \texttt{tol}\|\mathbf{x}^{(0)} - \mathbf{u}\|. \tag{10}$$

Specifically you are asked to deliver two programs:

(a) A MATLAB program (function)

`[xcomp,niter]=myjacobi(A, xexact, x0, tol)`

That gives the computed solution, `xcomp`, of $Ax = b$ using the Jacobi iterative method, and the number of iterations, `niter`, such that

$$\|\texttt{xcomp} - \texttt{xexact}\|_2 \leq \texttt{tol}\, \|\texttt{x0} - \texttt{xexact}\|_2.$$

The input variables are: the matrix `A`, the exact solution `xexact` (the independent term vector is `b=A*xexact`) and the initial guess for the solution `x0` (it can be set to zero).

(b) A MATLAB script (list of commands in an .m file, not a function) that, when executed, will run the Jacobi code for the values of $n$ above, producing the log-log plot in a figure (MATLAB command `loglog`) and a table with three columns: `n`, the number of iterations `niter` and the final value of the 2-norm of the absolute error.

Is this the expected number of iterations?

6. Repeat the previous part with Gauss-Seidel's method. How much faster is it?