Section 6.1: 6a and 6d

6. **Gaussian elimination gives the following solutions.**

   (a) $x_1 = -4$, $x_2 = -8$, $x_3 = -6$ with one row interchange required

   (b) $x_1 = \frac{22}{9}$, $x_2 = -\frac{4}{9}$, $x_3 = \frac{4}{3}, x_4 = 1$ with one row interchange required

   (c) $x_1 = 13$, $x_2 = 8$, $x_3 = 8, x_4 = 5$ with one row interchange required.

   (d) $x_1 = -1$, $x_2 = 2$, $x_3 = 0, x_4 = 1$ with one row interchange required.

REMARK: In 6a there is no solution.

Section 6.2: 8b and 8c.

8. The following row interchanges are required for these systems.

   (a) Interchange rows 1 and 2, and columns 1 and 3.

   (b) Interchange rows 1 and 2, and columns 1 and 2, then interchange rows 2 and 3.

   (c) Interchange rows 1 and 3, and columns 1 and 2, then interchange rows 2 and 3, and columns 2 and 3.

   (d) Interchange rows 1 and 2.

Section 6.3:

8. (a) Following the steps of Algorithm 6.1 with $m-1$ additional columns in the augmented matrix gives the following:

Reduction Steps 1–6:
Multiplications/Divisions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{1+(m+n-i)\} = \sum_{i=1}^{n-1}\{n(m+n+1)-(m+2n+1)i+i^2\}$$
$$= \frac{1}{2}mn^2 - \frac{1}{2}mn + \frac{1}{3}n^3 - \frac{1}{3}n$$

Additions/Subtractions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{m+n-i\} = \sum_{i=1}^{n-1}\{n(m+n)-(m+2n)i+i^2\}$$
$$= \frac{1}{2}mn^2 - \frac{1}{2}mn + \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

Backward Substitution Steps 8–9:
Multiplications/Divisions:

$$m\left[1+\sum_{i=1}^{n-1}(n-i+1)\right] = m\left[1+\frac{n(n+1)}{2}-1\right] = \frac{1}{2}mn^2 + \frac{1}{2}mn$$

Additions/Subtractions:

$$m\left[\sum_{i=1}^{n-1}(n-i)\right] = \frac{1}{2}mn^2 - \frac{1}{2}mn$$

Total:
Multiplications/Divisions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$
Additions/Subtractions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{2}n^2 - mn + \frac{1}{6}n$

(b) For the reduction phase: Multiplications/Divisions:

$$\sum_{i=1}^{n} \sum_{j=1 j\neq i}^{n} \left\{ 1 + \sum_{k=i+1}^{n+m} 1 \right\} = \sum_{i=1}^{n} \sum_{j=1 j\neq i}^{n} (m+n+1-i) = \sum_{i=1}^{n} \{(n-1)(m+n+1) - (n-1)i\}$$
$$= \frac{1}{2}n^3 + mn^2 - mn - \frac{1}{2}n$$

Additions/Subtractions:

$$\sum_{i=1}^{n} \sum_{j=1 j\neq i}^{n} \sum_{k=i+1}^{n+m} 1 = \sum_{i=1}^{n} \sum_{j=1 j\neq i}^{n} (n+m-i) = \sum_{i=1}^{n} \{(n-1)(m+n) - (n-1)i\}$$
$$= \frac{1}{2}n^3 + mn^2 - mn - n^2 + \frac{1}{2}n$$

Backward Substitution Steps:
Multiplications/Divisions:

$$\sum_{k=1}^{m} \sum_{i=1}^{n} 1 = mn$$

Additions/Subtractions: none
Totals:
Multiplications/Divisions: $\frac{1}{2}n^3 + mn^2 - \frac{1}{2}n$
Additions/Subtractions: $\frac{1}{2}n^3 + mn^2 - n^2 - mn + \frac{1}{2}n$

(c) When $m = n$ we have the following:

Gaussian Elimination
Multiplications/Divisions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n = \frac{4}{3}n^3 - \frac{1}{3}n$
Additions/Subtractions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{2}n^2 - mn + \frac{1}{6}n = \frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{1}{6}n$

Gauss-Jordan Elimination
Multiplications/Divisions: $\frac{1}{2}n^3 + mn^2 - \frac{1}{2}n = \frac{3}{2}n^3 - \frac{1}{2}n$
Additions/Subtractions: $\frac{1}{2}n^3 + mn^2 - n^2 - mn + \frac{1}{2}n = \frac{3}{2}n^3 - 2n^2 + \frac{1}{2}n$

(d) To find the inverse of the $n \times n$ matrix A:
INPUT $n \times n$ matrix $A = (a_{ij})$.
OUTPUT $n \times n$ matrix $B = A^{-1}$.
Step 1 Initialize the $n \times n$ matrix $B = (b_{ij})$ to

$$b_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases}$$

Step 2 For $i = 1, \ldots, n-1$ do Steps 3, 4, and 5.
Step 3 Let p be the smallest integer with $i \leq p \leq n$ and $a_{p,i} \neq 0$.
If no integer p can be found then
OUTPUT ('A is singular');
STOP.
Step 4 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.
Step 5 For $j = i+1, \ldots, n$ do Steps 6 through 9.

Step 6 Set $m_{ji} = a_{ji}/a_{ii}$.

Step 7 For $k = i+1, \ldots, n$
set $a_{jk} = a_{jk} - m_{ji}a_{ik}$; $a_{ij} = 0$.

Step 8 For $k = 1, \ldots, i-1$
set $b_{jk} = b_{jk} - m_{ji}b_{ik}$.

Step 9 Set $b_{ji} = -m_{ji}$.

Step 10 If $a_{nn} = 0$ then OUTPUT ('A is singular');
STOP.

Step 11 For $j = 1, \ldots, n$ do Steps 12, 13 and 14.

Step 12 Set $b_{nj} = b_{nj}/a_{nn}$.

Step 13 For $i = n-1, \ldots, j$
set $b_{ij} = \left(b_{ij} - \sum_{k=i+1}^{n} a_{ik}b_{kj}\right)/a_{ii}$.

Step 14 For $i = j-1, \ldots, 1$
set $b_{ij} = -\left[\sum_{k=i+1}^{n} a_{ik}b_{kj}\right]/a_{ii}$.

Step 15 OUTPUT (B);
STOP.

Reduction Steps 2–9:
Multiplications/Divisions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{1 + \sum_{k=i+1}^{n} 1 + \sum_{k=1}^{i-1} 1\right\} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{1+n-i+i-1\} = \frac{n^2(n-1)}{2}$$

Additions/Subtractions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\sum_{k=i+1}^{n} 1 + \sum_{k=1}^{i-1} 1\right\} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{n-i+i-1\} = \frac{n(n-1)^2}{2}$$

Backward Substitution Steps 11–14:
Multiplications/Divisions:

$$\sum_{j=1}^{n}\left\{1 + \sum_{i=j}^{n-1}\left\{1 + \sum_{k=i+1}^{n} 1\right\} + \sum_{i=1}^{j-1}\left\{1 + \sum_{k=i+1}^{n} 1\right\}\right\} = \sum_{j=1}^{n}\left\{1 + \sum_{i=j}^{n-1}(n+1-i) + \sum_{i=1}^{j-1}(n+1-i)\right\}$$

$$= \sum_{j=1}^{n}\left[1 + \sum_{i=1}^{n-1}(n+1-i)\right]$$

$$= \sum_{j=1}^{n}\frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$$

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Additions/Subtractions:

$$\sum_{j=1}^{n}\left\{\sum_{i=j}^{n-1}(1+n-i-1)+\sum_{i=1}^{j-1}(n-i-1)\right\}=\sum_{j=1}^{n}\sum_{i=1}^{n-1}(n-i)-j+1$$

$$=\sum_{j=1}^{n}\left[\frac{n(n-1)}{2}+1-j\right]$$

$$=\frac{n^2(n-1)}{2}+n-\frac{n(n+1)}{2}$$

$$=\frac{n^3}{2}-n^2+\frac{1}{2}n$$

Totals:
Multiplications/Divisions: $\frac{n^2(n-1)}{2}+\frac{n^2(n+1)}{2}=n^3$
Additions/Subtractions: $\frac{n(n-1)^2}{2}+\frac{n^3}{2}-n^2+\frac{1}{2}n=n^3-2n^2+n$

(e) Let $\left[A^{-1}\right]_{i,j}$ denote the entries of A^{-1}, for $1\le i,j\le n$. For each $i=1,\ldots,n$, we have

$$x_i=\sum_{j=1}^{n}[A^{-1}]_{i,j}b_j.$$

This requires n multiplications and $n-1$ additions for each i. The total number of computations is n^2 Multiplications/Divisions and n^2-n Additions/Subtractions.

(f) For m linear systems, we have mn^2 Multiplications/Divisions and $m(n^2-n)$ Additions/Subtractions.

(g)

	Gaussian Elimination (part a)		Inverting A and forming $A^{-1}b$	
n	Multiplications Divisions	Additions Subtractions	Multiplications Divisions	Additions Subtractions
3	$9m+8$	$6m+5$	$9m+27$	$6m+12$
10	$100m+330$	$90m+285$	$100m+1000$	$90m+810$
50	$2500m+41650$	$2450m+40425$	$2500m+125000$	$2450m+120050$
100	$10000m+333300$	$9900m+328350$	$10000m+1000000$	$9900m+980100$