# Iterative Techniques in Matrix Algebra

## Exercise Set 7.1, page 427

1. (a) We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.220153$(b) We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.477226$.

   (c) We have $||\mathbf{x}||_\infty = 2^k$ and $||\mathbf{x}||_2 = (1+4^k)^{1/2}$.

   (d) We have $||\mathbf{x}||_\infty = 4/(k+1)$ and $||\mathbf{x}||_2 = (16/(k+1)^2 + 4/k^4 + k^4e^{-2k})^{1/2}$.

2. (a) Since $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| \geq 0$ with equality only if $x_i = 0$ for all $i$, properties $(i)$ and $(ii)$ in Definition 7.1 hold.
   Also,

$$\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^n |\alpha x_i| = \sum_{i=1}^n |\alpha||x_i| = |\alpha|\sum_{i=1}^n |x_i| = |\alpha|\|\mathbf{x}\|_1,$$

   so property $(iii)$ holds.
   Finally,

$$\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^n |x_i+y_i| \leq \sum_{i=1}^n (|x_i|+|y_i|) = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1,$$

   so property $(iv)$ also holds.

   (b) (1a) 8.5 (1b) 10 (1c) $|\sin k| + |\cos k| + e^k$ (1d) $4/(k+1) + 2/k^2 + k^2e^{-k}$

   (c) We have

$$\begin{aligned}\|\mathbf{x}\|_1^2 = \left(\sum_{i=1}^n |x_i|\right)^2 &= (|x_1| + |x_2| + \cdots + |x_n|)^2 \\ &\geq |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = \sum_{i=1}^n |x_i|^2 = \|\mathbf{x}\|_2^2.\end{aligned}$$

   Thus, $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$.

3. (a) We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0,0,0)^t$.  (b) We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0,1,3)^t$.

   (c) We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0,0,\frac{1}{2})^t$.  (d) We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (1,-1,1)^t$.

4. The $||\cdot||_\infty$ norms are as follows:

   (a) 25  (b) 16  (c) 4  (d) 12

5. (a) We have $||\mathbf{x}-\hat{\mathbf{x}}||_\infty = 8.57 \times 10^{-4}$ and $||A\hat{\mathbf{x}}-\mathbf{b}||_\infty = 2.06 \times 10^{-4}$.
   (b) We have $||\mathbf{x}-\hat{\mathbf{x}}||_\infty = 0.90$ and $||A\hat{\mathbf{x}}-\mathbf{b}||_\infty = 0.27$.
   (c) We have $||\mathbf{x}-\hat{\mathbf{x}}||_\infty = 0.5$ and $||A\hat{\mathbf{x}}-\mathbf{b}||_\infty = 0.3$.
   (d) We have $||\mathbf{x}-\hat{\mathbf{x}}||_\infty = 6.55 \times 10^{-2}$, and $||A\hat{\mathbf{x}}-\mathbf{b}||_\infty = 0.32$.

6. The $||\cdot||_\infty$ norms are as follows:

   (a) 16  (b) 25  (c) 4  (d) 12

7. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $\|AB\|_{\textcircled{$\infty$}} = 2$, but $\|A\|_{\textcircled{$\infty$}} \cdot \|B\|_{\textcircled{$\infty$}} = 1$.

8. Showing properties (i) – (iv) of Definition 7.8 is similar to the proof in Exercise 2a. To show property (v),

$$
\begin{aligned}
\|AB\|_{\textcircled{1}} &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{ik}b_{kj}\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}||b_{kj}| \\
&= \sum_{i=1}^{n}\left\{\sum_{k=1}^{n}|a_{ik}|\sum_{j=1}^{n}|b_{kj}|\right\} \le \sum_{i=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}|\right)\left(\sum_{k=1}^{n}\sum_{j=1}^{n}|b_{kj}|\right) \\
&= \left(\sum_{i=1}^{n}\sum_{k=1}^{n}|a_{ik}|\right)\|B\|_{\textcircled{1}} = \|A\|_{\textcircled{1}}\|B\|_{\textcircled{1}}.
\end{aligned}
$$

The norms of the matrices in Exercise 4 are (4a) 26, (4b) 26, (4c) 10, and (4d) 28.

9. (a) Showing properties (*i*)-(*iv*) of Definition 7.8 is straight-forward. Property (*v*) is shown as follows:

$$\begin{aligned}
\|AB\|_F^2 &= \left(\sum_{i=1}^n \sum_{j=1}^n \left|\sum_{k=1}^n a_{ik}b_{kj}\right|^2\right) \\
&\le \left(\sum_{i=1}^n \sum_{j=1}^n \left(\sum_{k=1}^n |a_{ik}|^2 \sum_{k=1}^n |b_{kj}|^2\right)\right) \qquad \text{by Theorem 7.3} \\
&= \sum_{i=1}^n \sum_{k=1}^n \left[|a_{ik}|^2 \left(\sum_{j=1}^n \sum_{k=1}^n |b_{kj}|^2\right)\right] \\
&= \sum_{i=1}^n \sum_{k=1}^n |a_{ik}|^2 \|B\|_F^2 = \|B\|_F^2 \sum_{i=1}^n \sum_{k=1}^n |a_{ik}|^2 = \|B\|_F^2 \|A\|_F^2 = \|A\|_F^2 \|B\|_F^2.
\end{aligned}$$

(b) We have

(4a) $\|A\|_F = \sqrt{326}$

(4b) $\|A\|_F = \sqrt{326}$

(4c) $\|A\|_F = 4$

(4d) $\|A\|_F = \sqrt{148}$.

(c)

$$\begin{aligned}
\|A\|_2^2 &= \max_{\|\mathbf{x}\|_2=1} \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j\right)^2 \le \max_{\|\mathbf{x}\|_2=1} \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}||x_j|\right)^2 \\
&\le \max_{\|\mathbf{x}\|_2=1} \sum_{i=1}^n \left[\left(\sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n |x_j|^2\right)^{\frac{1}{2}}\right]^2 = \max_{\|\mathbf{x}\|_2=1} \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^2\right)\left(\sum_{j=1}^n |x_j|^2\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = \|A\|_F^2
\end{aligned}$$

Let $j$ be fixed and define

$$x_k = \begin{cases} 0, & \text{if } k \neq j \\ 1, & \text{if } k = j. \end{cases}$$

Then $A\mathbf{x} = (a_{1j}, a_{2j}, \ldots, a_{nj})^t$, so

$$\|A\|_2^2 \ge \|A\mathbf{x}\|_2^2 \ge \sum_{i=1}^n |a_{ij}|^2.$$

Thus,

$$\|A\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = \sum_{j=1}^n \sum_{i=1}^n |a_{ij}|^2 \le \sum_{j=1}^n \|A\|_2^2 = n\|A\|_2^2.$$

Hence, $\|A\|_2 \le \|A\|_F \le \sqrt{n}\|A\|_2$.

10. We have

$$\|A\mathbf{x}\|_2^2 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij}x_j \right|^2 \leq \sum_{i=1}^n \left( \sum_{j=1}^n |a_{ij}||x_j| \right)^2 .$$

Using the Cauchy–Buniakowsky–Schwarz inequality gives

$$\|A\mathbf{x}\|_2^2 \leq \sum_{i=1}^n \left( \left( \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}} \right)^2 = \sum_{i=1}^n \left( \sum_{j=1}^n |a_{ij}|^2 \right) \|\mathbf{x}\|_2^2 = \|A\|_F^2 \|\mathbf{x}\|_2^2.$$

Thus, $\|A\mathbf{x}\|_2 \leq \|A\|_F \|\mathbf{x}\|_2$.

11. That $\|\mathbf{x}\| \geq 0$ follows easily. That $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ follows from the definition of positive definite. In addition,

$$\|\alpha\mathbf{x}\| = \left[ \left(\alpha\mathbf{x}^t\right) S(\alpha\mathbf{x}) \right]^{\frac{1}{2}} = \left[\alpha^2 \mathbf{x}^t S\mathbf{x}\right]^{\frac{1}{2}} = |\alpha| \left(\mathbf{x}^t S\mathbf{x}\right)^{\frac{1}{2}} = |\alpha|\|\mathbf{x}\|.$$

From Cholesky's factorization, let $S = LL^t$. Then

$$\begin{aligned}
\mathbf{x}^t S\mathbf{y} &= \mathbf{x}^t LL^t\mathbf{y} = \left(L^t\mathbf{x}\right)^t \left(L^t\mathbf{y}\right) \\
&\leq \left[ \left(L^t\mathbf{x}\right)^t \left(L^t\mathbf{x}\right) \right]^{1/2} \left[ \left(L^t\mathbf{y}\right)^t \left(L^t\mathbf{y}\right) \right]^{1/2} \\
&= \left(\mathbf{x}^t LL^t\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t LL^t\mathbf{y}\right)^{1/2} = \left(\mathbf{x}^t S\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= \left[ (\mathbf{x}+\mathbf{y})^t S(\mathbf{x}+\mathbf{y}) \right] = \left[ \mathbf{x}^t S\mathbf{x} + \mathbf{y}^t S\mathbf{x} + \mathbf{x}^t S\mathbf{y} + \mathbf{y}^t S\mathbf{y} \right] \\
&\leq \mathbf{x}^t S\mathbf{x} + 2\left(\mathbf{x}^t S\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2} + \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2} \\
&= \mathbf{x}^t S\mathbf{x} + 2\|\mathbf{x}\|\|\mathbf{y}\| + \mathbf{y}^t S\mathbf{y} = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

This demonstrates properties $(i)$ – $(iv)$ of Definition 7.1.

12. Since $\|\mathbf{x}\|' = 0$ implies $\|S\mathbf{x}\| = 0$, we have $S\mathbf{x} = \mathbf{0}$. Since $S$ is nonsingular, $\mathbf{x} = \mathbf{0}$. Also,

$$\|\mathbf{x}+\mathbf{y}\|' = \|S(\mathbf{x}+\mathbf{y})\| = \|S\mathbf{x} + S\mathbf{y}\| \leq \|S\mathbf{x}\| + \|S\mathbf{y}\| = \|\mathbf{x}\|' + \|\mathbf{y}\|'$$

and

$$\|\alpha\mathbf{x}\|' = \|S(\alpha\mathbf{x})\| = |\alpha| \; \|S\mathbf{x}\| = |\alpha| \; \|\mathbf{x}\|'.$$

13. It is not difficult to show that $(i)$ holds. If $\|A\| = 0$, then $\|A\mathbf{x}\| = 0$ for all vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 1$. Using $\mathbf{x} = (1, 0, \ldots, 0)^t$, $\mathbf{x} = (0, 1, 0, \ldots, 0)^t, \ldots,$ and $\mathbf{x} = (0, \ldots, 0, 1)^t$ successively implies that each column of $A$ is zero. Thus, $\|A\| = 0$ if and only if $A = 0$. Moreover,

$$\begin{aligned}
\|\alpha A\| &= \max_{\|\mathbf{x}\|=1} \|(\alpha A\mathbf{x})\| = |\alpha| \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = |\alpha| \cdot \|A\|, \\
\|A+B\| &= \max_{\|\mathbf{x}\|=1} \|(A+B)\mathbf{x}\| \leq \max_{\|\mathbf{x}\|=1} (\|A\mathbf{x}\| + \|B\mathbf{x}\|),
\end{aligned}$$