Exercise Set 7.2, page 435

1. (a) The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 1$ has the eigenvector $\mathbf{x}_2 = (1, 1)^t$.

   (b) The eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, (1+\sqrt{5})/2\right)^t$, and the eigenvalue $\lambda_2 = \frac{1-\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, (1-\sqrt{5})/2\right)^t$.

   (c) The eigenvalue $\lambda_1 = \frac{1}{2}$ has the eigenvector $\mathbf{x}_1 = (1, 1)^t$, and the eigenvalue $\lambda_2 = -\frac{1}{2}$ has the eigenvector $\mathbf{x}_2 = (1, -1)^t$.

   (d) The eigenvalue $\lambda_1 = 1$ has the eigenvector $\mathbf{x}_1 = (1, -1, 0)^t$, and the eigenvalue $\lambda_2 = \lambda_3 = 3$ has the eigenvectors $\mathbf{x}_2 = (1, 1, 0)^t$ and $\mathbf{x}_3 = (1, 1, 1)^t$.

   (e) The eigenvalue $\lambda_1 = 1$ has the eigenvector $\mathbf{x}_1 = (1, 1, 4)^t$, the eigenvalue $\lambda_2 = -1$ has the eigenvector $\mathbf{x}_2 = (1, 0, 0)^t$, and the eigenvalue $\lambda_3 = 3$ has the eigenvector $\mathbf{x}_3 = (1, 2, 0)^t$.

   (f) The eigenvalue $\lambda_1 = \lambda_2 = 1$ has the eigenvectors $\mathbf{x}_1 = (-1, 0, 1)^t$ and $\mathbf{x}_2 = (-1, 1, 0)^t$, and the eigenvalue $\lambda_3 = 5$ has the eigenvector $\mathbf{x} = (1, 2, 1)^t$.

2. (a) The eigenvalue $\lambda_1 = 0$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = -1$ has the eigenvector $\mathbf{x}_2 = (1, -2)^t$.

   (b) The eigenvalue $\lambda_1 = (3 + \sqrt{7}i)/2$ has the eigenvector $\mathbf{x}_1 = (1 - \sqrt{7}i)/2, 1)^t$, and the eigenvalue $\lambda_2 = (3 - \sqrt{7}i)/2$ has the eigenvector $\mathbf{x}_2 = (1 + \sqrt{7}i)/2, 1)^t$.

   (c) The eigenvalue $\lambda_1 = -1$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 4$ has the eigenvector $\mathbf{x}_2 = (4, 1)^t$.

   (d) The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (-1, 1, 2)^t$, the eigenvalue $\lambda_2 = 4$ has the eigenvector $\mathbf{x}_2 = (0, 1, 2)^t$, and the eigenvalue $\lambda_3 = -2$ has the eigenvector $\mathbf{x} = (-3, 8, 1)^t$.

   (e) The eigenvalue $\lambda_1 = \lambda_2 = 1/2$ has the eigenvector $\mathbf{x}_1 = (0, 5, 12)^t$, and the eigenvalue $\lambda_3 = -1/3$ has the eigenvector $\mathbf{x}_3 = (0, 0, 1)^t$.

   (f) The eigenvalue $\lambda_1 = 2 + 2i$ has the eigenvector $\mathbf{x}_1 = (0, -2i, 1)^t$, the eigenvalue $\lambda_2 = 2 - 2i$ has the eigenvector $\mathbf{x}_2 = (0, 2i, 1)^t$, and the eigenvalue $\lambda_3 = 2$ has the eigenvector $\mathbf{x}_3 = (1, 0, 0)^t$.

3. The spectral radii for the matrices in Exercise 1 are;

   (a) 3 (b) $\frac{1+\sqrt{5}}{2}$ (c) 1/2 (d) 3 (e) 7 (f) 5

4. The spectral radii for the matrices in Exercise 2 are:

(a) 1 (b) 2 (c) 4 (d) 4 (e) 1/2 (f) $2\sqrt{2}$

5. Only the matrix in 1(c) is convergent.

6. Only the matrix in 2(e) is convergent.

7. The $||\cdot||_2$ norms for the matrices in Exercise 1 are:

(a) 3 (b) 1.618034 (c) 0.5 (d) 3 (e) 8.224257 (f) 5.203527

8. The $||\cdot||_2$ norms for the matrices in Exercise 1 are:

(a) 3.162278 (b) 2.828427 (c) 5.036796 (d) 5.601152 (e) 2.896954 (f) 4.701562

9. Since

$$A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{2^k-1}{2^{k+1}} & 2^{-k} \end{bmatrix}, \quad \text{we have} \quad \lim_{k\to\infty} A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Also,

$$A_2^k = \begin{bmatrix} 2^{-k} & 0 \\ \frac{16k}{2^{k-1}} & 2^{-k} \end{bmatrix}, \quad \text{so} \quad \lim_{k\to\infty} A_2^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

10. If $\mathbf{y}$ is an eigenvector, then $\mathbf{x} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$ is also an eigenvector.

11. Let A be an $n \times n$ matrix. Expanding across the first row gives the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \sum_{j=2}^{n}(-1)^{j+1}a_{1j}M_{1j}.$$

The determinants M_{1j} are of the form

$$M_{1j} = \det \begin{bmatrix} a_{21} & a_{22}-\lambda & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,j-1}-\lambda & a_{j-1,j+1} & \cdots & a_{j-1,n} \\ a_{j,1} & a_{j,2} & \cdots & a_{j,j-1} & a_{j,j+1} & \cdots & a_{j,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,j-1} & a_{j+1,j+1}-\lambda & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn}-\lambda \end{bmatrix},$$

for $j = 2, \ldots, n$. Note that each M_{1j} has $n-2$ entries of the form $a_{ii} - \lambda$. Thus,

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \{\text{terms of degree } n-2 \text{ or less}\}.$$

Since

$$M_{11} = \det \begin{bmatrix} a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{32} & a_{33}-\lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n} \\ a_{n2} & \cdots & \cdots & a_{n,n-1} & a_{nn}-\lambda \end{bmatrix}$$

is of the same form as $\det(A - \lambda I)$, the same argument can be repeatedly applied to determine

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + \{\text{terms of degree } n-2 \text{ or less in } \lambda\}.$$

Thus, $p(\lambda)$ is a polynomial of degree n.

12. (a) $P(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda) = \det(A - \lambda I)$, so $P(0) = \lambda_1 \cdots \lambda_n = \det A$.
    (b) A singular if and only if $\det A = 0$, which is equivalent to at least one of λ_i being 0.

13. (a) $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$
    (b) If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$, and by induction, $A^k\mathbf{x} = \lambda^k\mathbf{x}$.
    (c) If $A\mathbf{x} = \lambda\mathbf{x}$ and A^{-1} exists, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. By Exercise 8 (b), $\lambda \neq 0$, so $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$.
    (d) Since $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$, we have $(A^{-1})^2\mathbf{x} = \frac{1}{\lambda}A^{-1}\mathbf{x} = \frac{1}{\lambda^2}\mathbf{x}$. Mathematical induction gives

    $$(A^{-1})^k\mathbf{x} = \frac{1}{\lambda^k}\mathbf{x}.$$

    (e) If $A\mathbf{x} = \lambda\mathbf{x}$, then

    $$q(A)\mathbf{x} = q_0\mathbf{x} + q_1A\mathbf{x} + \ldots + q_kA^k\mathbf{x} = q_0\mathbf{x} + q_1\lambda\mathbf{x} + \ldots + q_k\lambda^k\mathbf{x} = q(\lambda)\mathbf{x}.$$

    (f) Let $A - \alpha I$ be nonsingular. Since $A\mathbf{x} = \lambda\mathbf{x}$,

    $$(A - \alpha I)\mathbf{x} = A\mathbf{x} - \alpha I\mathbf{x} = \lambda\mathbf{x} - \alpha\mathbf{x} = (\lambda - \alpha)\mathbf{x}.$$

    Thus,

    $$\frac{1}{\lambda - \alpha}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}.$$

14. Since $A^tA = A^2$ and $A\mathbf{x} = \lambda\mathbf{x}$, we have $A^2\mathbf{x} = \lambda^2\mathbf{x}$. Thus, $\rho(A^tA) = \rho(A^2) = [\rho(A)]^2$ and $\|A\|_2 = [\rho(A^tA)]^{\frac{1}{2}} = \rho(A)$.

15. (a) We have the real eigenvalue $\lambda = 1$ with the eigenvector $\mathbf{x} = (6, 3, 1)^t$.
    (b) Choose any multiple of the vector $(6, 3, 1)^t$.

16. For

    $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

    we have $\rho(A) = \rho(B) = 1$ and $\rho(A + B) = 3$.

17. Let $A\mathbf{x} = \lambda\mathbf{x}$. Then $|\lambda|\,\|\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\|\,\|\mathbf{x}\|$, which implies $|\lambda| \leq \|A\|$. Also, $(1/\lambda)\mathbf{x} = A^{-1}\mathbf{x}$ so $1/|\lambda| \leq \|A^{-1}\|$ and $\|A^{-1}\|^{-1} \leq |\lambda|$.

Exercise Set 7.4, page 461

1. The $\|\cdot\|_\infty$ condition number is:

   (a) 50 (b) 241.37 (c) 600,002 (d) 339,866

2. The $\|\cdot\|_\infty$ condition numbers are:

   (a) 12.24012756 (b) 12.24012756 (c) 12 (d) 198.17

3. We have

	$\|\mathbf{x}-\hat{\mathbf{x}}\|_\infty$	$K_\infty(A)\|\mathbf{b}-A\hat{\mathbf{x}}\|_\infty/\|A\|_\infty$
(a)	8.571429×10^{-4}	1.238095×10^{-2}
(b)	0.1	3.832060
(c)	0.04	0.8
(d)	20	1.152440×10^{5}

4. We have

	$\|\mathbf{x}-\hat{\mathbf{x}}\|_\infty$	$K_\infty(A)\|\mathbf{b}-A\hat{\mathbf{x}}\|_\infty/\|A\|_\infty$
(a)	20	65.03241
(b)	0.02	720.5764
(c)	0.1	3.727412×10^{-1}
(d)	6.551700×10^{-2}	9.059201

5. Gaussian elimination and iterative refinement give the following results.

   (a) (i) $(-10.0, 1.01)^t$, (ii) $(10.0, 1.00)^t$

   (b) (i) $(12.0, 0.499, -1.98)^t$, (ii) $(1.00, 0.500, -1.00)^t$

   (c) (i) $(0.185, 0.0103, -0.0200, -1.12)^t$, (ii) $(0.177, 0.0127, -0.0207, -1.18)^t$

   (d) (i) $(0.799, -3.12, 0.151, 4.56)^t$, (ii) $(0.758, -3.00, 0.159, 4.30)^t$

6. Gaussian elimination and iterative refinement give the following results.

   (a) $(12.00, 0.9990)^t$, $(10.00, 1.000)^t$

   (b) $(1.200, 0.5002, -1.380)^t$, $(1.000, 0.5000, -0.9998)^t$

   (c) $(0.1756, 0.01305, -0.02075, -1.192)^t$, $(0.1768, 0.01269, -0.02065, -1.182)^t$

   (d) $(0.7963, -3.152, 0.1705, 4.615)^t$, $(0.7889, -3.128, 0.1678, 4.561)^t$

7. The matrix is ill-conditioned since $K_\infty = 60002$. We have $\tilde{\mathbf{x}} = (-1.0000, 2.0000)^t$.

8. The matrix A is ill-conditioned since $K_\infty(A) = 600,002$ and $\hat{\mathbf{x}} = (1.818192, 0.5909091)^t$