

Homework #3

Programming problem

$$A = [a_{ij}] \quad a_{ij} = \begin{cases} -\frac{1}{h^2} & |i-j|=1 \\ \frac{2}{h^2} & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$A = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ 0 & & -1 & 2 \end{bmatrix}$$

1. $u_i^{(k)} = \sin\left(\frac{\pi k i}{n+1}\right)$

For $k=1, \dots, n$ is an eigenvector of A

($i \neq 1, n$) Take the i -th component of $Au^{(k)}$

$$(Au^{(k)})_i = \frac{1}{h^2} \left(-\sin \frac{\pi k (i-1)}{n+1} + 2 \sin \frac{\pi k i}{n+1} - \sin \frac{\pi k (i+1)}{n+1} \right)$$

$$\stackrel{?}{=} \lambda_k u_i^{(k)} = \lambda_k \sin \frac{\pi k i}{n+1}$$

let us use that

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

Then

$$(Au^{(k)})_i = \frac{1}{h^2} \left(2 \sin \frac{\pi k i}{n+1} - 2 \sin \frac{\pi k i}{n+1} \cdot \cos \frac{\pi k}{n+1} \right) =$$

$$= \frac{1}{h^2} 2 \sin \frac{\pi k i}{n+1} \left(1 - \cos \frac{\pi k}{n+1} \right) \Rightarrow$$

$$\Rightarrow \boxed{\lambda_k = \frac{2}{h^2} \left(1 - \cos \frac{\pi k}{n+1} \right) \quad k = 1, \dots, n}$$

let us check the first and the last component

$i=1$

$$(Au^{(k)})_1 = \frac{1}{h^2} \left(2 \sin \frac{\pi k}{n+1} - \sin \frac{\pi k 2}{n+1} \right) \stackrel{?}{=} \lambda_k \sin \frac{\pi k}{n+1}$$

$$\frac{1}{h^2} \left(2 \sin \frac{\pi k}{n+1} - 2 \sin \frac{\pi k}{n+1} \cos \frac{\pi k}{n+1} \right)$$

$$\frac{1}{h^2} 2 \left(1 - \cos \frac{\pi k}{n+1} \right) \sin \frac{\pi k}{n+1} = \lambda_k \sin \frac{\pi k}{n+1} \quad \checkmark$$

$i=n$

$$(Au^{(k)})_n = \frac{1}{h^2} \left(-\sin \frac{\pi k(n-1)}{n+1} + 2 \sin \frac{\pi k n}{n+1} \right) \stackrel{?}{=} \lambda_k \sin \frac{\pi k}{n+1} \cdot n$$

$$= \frac{1}{h^2} \left(2 \sin \frac{\pi k n}{n+1} - \sin \frac{\pi k n}{n+1} \cos \frac{\pi k}{n+1} + \cos \frac{\pi k}{n+1} \sin \frac{\pi k}{n+1} \right) =$$

$$\stackrel{?}{=} \frac{2}{h^2} \left(1 - \cos \frac{\pi k}{n+1} \right) \sin \frac{\pi k}{n+1} n$$

This is true if

$$-\sin \frac{\pi K}{n+1} n \cos \frac{\pi K}{n+1} + \cos \frac{\pi K}{n+1} n \sin \frac{\pi K}{n+1} = -2 \cos \frac{\pi K}{n+1} \sin \frac{\pi K}{n+1} n$$

$$\Rightarrow \cos \frac{\pi K}{n+1} \sin \frac{\pi K}{n+1} n + \cos \frac{\pi K}{n+1} n \sin \frac{\pi K}{n+1} =$$

$$= \sin \left(\frac{\pi K}{n+1} n + \frac{\pi K}{n+1} \right) = \sin \left(\frac{\pi K(n+1)}{(n+1)} \right) = 0 \quad \checkmark$$

Therefore, the vectors $u^{(K)}$ with components

$$\left(u_i^{(K)} = \sin \left(\frac{\pi K}{n+1} i \right) \quad i=1, \dots, n \quad K=1, \dots, n \right)$$

are e-vectors of A with e-value

$$\left(\lambda_K = \frac{2}{h^2} \left(1 - \cos \frac{\pi K}{n+1} \right) \right)$$

② The Jacobi iteration is

$$A = L + D + U$$

$$L = \begin{bmatrix} 0 & & \\ -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} \quad \frac{1}{h^2} \cdot U = L^T \quad D = \frac{1}{h^2} \begin{bmatrix} 2 & & \\ & 0 & \\ & & 2 \end{bmatrix}$$

$$(L + D + U)x = b$$

$$Dx^{(k+1)} = -(L+U)x^{(k)} + b$$

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

In component

$$i=1 \quad x_1^{(k+1)} = \frac{1}{2} x_2^{(k)} + \frac{h^2}{2} b_1$$

$$i=2:n-1 \quad x_i^{(k+1)} = \frac{1}{2} (x_{i-1}^{(k)} + x_{i+1}^{(k)}) + \frac{h^2}{2} b_i$$

$$i=n \quad x_n^{(k+1)} = \frac{1}{2} x_{n-1}^{(k)} + \frac{h^2}{2} b_n$$

• The e-vectors of A : $u^{(k)}$ are also e-vectors of $T_J = -D^{-1}(L+U)$:

Proof

$$\begin{aligned} T_J &= -D^{-1}(L+U) = -D^{-1}(A-D) = -D^{-1}A + I = \\ &= -\frac{h^2}{2}A + I \quad \text{because} \quad D = \frac{2}{h^2}I_n \end{aligned}$$

$$\begin{aligned} \text{Then} \quad T_J u^{(k)} &= \left(-\frac{h^2}{2}A + I\right) u^{(k)} = -\frac{h^2}{2} \lambda_k u^{(k)} + u^{(k)} = \\ &= \left(-\frac{h^2}{2} \lambda_k + 1\right) u^{(k)} \end{aligned}$$

using $\lambda_k = \frac{2}{h^2} \left(1 - \cos \frac{\pi k}{n+1}\right)$ we get

that the e-values of T_J are

$$z_k = -\frac{h^2}{2} \lambda_k + 1 = -\frac{h^2}{2} \frac{2}{h^2} \left(1 - \cos \frac{\pi k}{n+1}\right) + 1 =$$

$$\boxed{z_k = \cos \frac{\pi k}{n+1} \quad \sigma(T_J) = \{z_k\}_{k=1}^n}$$

③ The spectral radius of T_J

$$\rho = \max_k |z_k| = \max_k \left| \cos \frac{\pi k}{n+1} \right| = \cos \frac{\pi}{n+1} \approx 1 - \frac{1}{2} \left(\frac{\pi}{n+1} \right)^2$$

$$1) \quad \|x^{(k)} - x^*\|_2 \leq \rho(\mathcal{T})^k \|x^{(0)} - x^*\|_2$$

Take $x^{(0)} = 0$

Then the relative error is

$$\frac{\|x^{(k)} - x^*\|_2}{\|x^*\|_2} \leq \rho(\mathcal{T})^k$$

We want

$$\rho^k \leq \epsilon$$

Taking $\log$ s

$$k \log \rho \leq \log \epsilon \quad \cancel{k \log \rho \leq \log \epsilon}$$

$\log \rho < 0$ (because $\rho < 1$) and also $\log \epsilon < 0$

Therefore

$$k |\log \rho| \geq |\log \epsilon| \Rightarrow \boxed{k \geq \frac{|\log \epsilon|}{|\log \rho|}}$$

let us say that $\rho = 10^{-2}$ and $\epsilon = 10^{-10}$
then

$$k \geq \frac{10}{2} = 5 \quad \text{Five or more iterations}$$

are needed to reduce the error to 10^{-10}