HW 4 Solutions

September 5, 2011

Exercise 8.2.2a
The goal is to find a_0, a_1, a_2 such that the error

$$E(a_0, a_1, a_2) = \int_{-1}^{1} \left(f(x) - \sum_{k=0}^{2} a_k P_k(x) \right)^2 \, dx$$

is minimized, and $P_k(x)$ are the Legendre polynomials given on page 501, that is

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= x^2 - \frac{1}{3}. \end{aligned}$$

It follows from the explanation on page 499 that

$$a_j = \frac{1}{\alpha_j} \int_{-1}^{1} f(x) P_j(x) \, dx,$$

where

$$\alpha_j = \int_{-1}^{1} P_j(x)^2 \, dx.$$

Using this we can easily compute

$$\begin{aligned} \alpha_0 &= 2 \\ \alpha_1 &= \frac{2}{3} \\ \alpha_2 &= \frac{8}{45} \end{aligned}$$

In this exercise $f(x) = x^2 - 2x + 3$, so

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^{1} x^2 - 2x + 3 \, dx \\ &= \frac{10}{3} \\ a_1 &= \frac{3}{2} \int_{-1}^{1} \left(x^2 - 2x + 3\right) x \, dx \\ &= -2 \\ a_2 &= \frac{45}{8} \int_{-1}^{1} \left(x^2 - 2x + 3\right) \left(x^2 - \frac{1}{3}\right) \, dx \\ &= 1, \end{aligned}$$

yielding the least squares approximating polynomial

$$\sum_{k=0}^{2} a_k P_k(x),$$

with a_k and $P_k(x)$as above.

Exercise 8.2.2c

The function for this problem is $f(x) = \frac{1}{x} + 2$. The solution to this problem follows relatively easily from the work done in the previous problem.

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} \frac{1}{x+2}\, dx \\ &= \frac{\log(3)}{2} \approx .55 \\ a_1 &= \frac{3}{2}\int_{-1}^{1} \frac{1}{x+2} x\, dx \\ &= 3 - \log(27) \approx -.30 \\ a_2 &= \frac{45}{8}\int_{-1}^{1} \frac{1}{x+2}\left(x^2 - \frac{1}{3}\right) dx \\ &\approx .16 \end{aligned}$$

This yields the least squares approximating polynomial

$$\sum_{k=0}^{2} a_k P_k(x),$$

as above.