Math. 108A. Homework 3:

Chapte 2, Number 5.

Show that F^∞ is finite-dimensional.

Proof: If F^∞ is finite-dimensional, $F^\infty = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ for some finite list of vectors $\vec{v}_1, \dots, \vec{v}_m$ in F^∞ (definition, p. 22). By the Comparison Theorem (Theorem 2.6), if $(\vec{u}_1, \dots, \vec{u}_{m+1})$ is a list of vectors in F^∞, then $(\vec{u}_1, \dots, \vec{u}_{m+1})$ is linearly dependent. But

$$\vec{u}_1 = (1, 0, \dots, 0, \dots), \quad \vec{u}_2 = (0, 1, \dots, 0, \dots),$$

$$\dots, \quad \vec{u}_{m+1} = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ m+1}}{1}, 0, \dots)$$

is linearly independent, because

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_{m+1}\vec{u}_{m+1} = 0$$

$$\Rightarrow (a_1, a_2, \dots, a_m, a_{m+1}, 0, \dots) = 0$$

$$\Rightarrow a_1 = 0, \ a_2 = 0, \dots, a_m = 0, \ a_{m+1} = 0.$$

This contradicts the fact that $(\vec{u}_1, \dots, \vec{u}_{m+1})$ is linearly dependent. So F^∞ is infinite-dimensional.

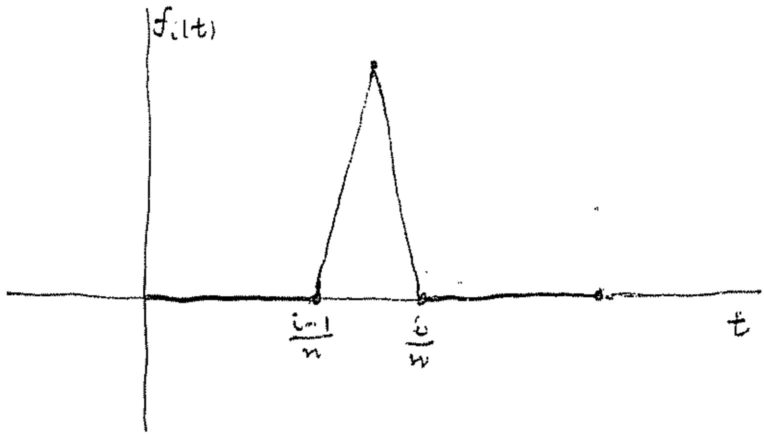
Chapter 2, Number 6.

Prove that the real vector space V of continuous functions $f : [0,1] \longrightarrow \mathbb{R}$ is infinite-dimensional.

Proof: Suppose V is finite-dimensional. Then $V = \operatorname{span}(g_1, \dots, g_m)$ for some finite list of functions $(g_1, \dots, g_m)$. Let $n > m$. We will obtain a contradiction by constructing a list of continuous functions $(f_1, \dots, f_n)$ which are linearly independent.

Let $f_i : [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that

$f_i \equiv 0$ on $[0, i-1]$ and $[i, n]$ $\qquad f_i\left(\frac{2i-1}{2n}\right) = 1$



$$f_i(t) = \begin{cases} 0, & t \in [0, i-1], \\ 2n\left(t - \frac{i-1}{n}\right), & t \in \left[i-1, \frac{2i-1}{2n}\right], \\ -2n\left(t - \frac{i}{n}\right), & t \in \left[\frac{2i-1}{n}, i\right], \\ 0, & t \in [i, n] \end{cases}$$

Then $f_i\left(\frac{2j-1}{n}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Thus $a_1 f_1 + \cdots + a_j f_j + \cdots + a_n f_n = 0$

$\Rightarrow \quad a_1 f_1\left(\frac{2i-1}{n}\right) + \cdots + a_j f_j\left(\frac{2i-1}{n}\right) + \cdots + a_n f_n\left(\frac{2i-1}{n}\right) = 0$

$\Rightarrow \quad a_i = 0$ for each i.

Thus $(f_1, \ldots, f_n)$ is linearly independent. This is a contradiction, so V must be infinite-dimensional.

Chapter 2, Number 8

$U = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2,\ x_3 = 7x_4 \}$

Solve for x_1 and x_3

$$\begin{aligned} x_1 &= -3x_2 \\ x_2 &= x_2 \\ x_3 &= -7x_4 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \qquad \vec{x} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

Basis: $\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

Chapter 2, Number 13,

Suppose U and W are subspaces of $\mathbb{R}^9$, $\dim U = \dim W = 5$

$U + W = \mathbb{R}^9$. Then $U \cap W \neq \{0\}$.

Then, by Theorem 2.18, $\dim U \cap W = \dim U + \dim W - \dim(U + W)$

$= 5 + 5 - 9 = 1$. Hence $U \cap W \neq \{0\}$.