Math. 108A Homework 5 Due Thursday, July 24, 2008

3.1. Let $(\vec{v}_1)$ be a basis for V. Then $T\vec{v}_1 = a\vec{v}_1$,

for some $a \in F$

If $\vec{v} \in V$, $\vec{v} = c\vec{v}_1$ and $T\vec{v} = T(c\vec{v}_1) = cT(\vec{v}_1) = ca\vec{v}_1$

$= a(c\vec{v}_1) = a\vec{v}$.

3.2. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{y^3 - 4x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then

$$f(ax,ay) = \begin{cases} \dfrac{a^3y^3 - 4a^3x^2y}{a^2(x^2+y^2)} = a\dfrac{y^3 - 4x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Thus $f(a\vec{v}) = af(\vec{v})$.

If $y = \pm 2x$, $y^2 - 4x^2 = 4x^2 - 4x^2 = 0$, so

$y = \pm 2x \Rightarrow f(x,y) = 0$

In particular, $f(1,2) = 0 = f(-1,2)$.

On the other hand, $f(0,y) = y$, so

$f((1,2)+(-1,2)) = f(0,4) = 4 \neq f(1,2) + f(-1,2)$

so f is NOT linear.

3.5. Suppose $a_1 T\vec{v}_1 + \cdots + a_n T\vec{v}_n = \vec{0}$. Then by linearity $T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = \vec{0} = T(\vec{0})$. Since T is injective $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$. Since $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent, $a_1 = a_2 = \cdots = a_n = 0$. Hence $(T\vec{v}_1, \ldots, T\vec{v}_n)$ is linearly independent.

3.7. Suppose $\vec{w} \in W$. Since T is surjective $\exists\, \vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Hence

$$\vec{w} = T(\vec{v}) = T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)$$
$$= a_1 T\vec{v}_1 + \cdots + a_n T\vec{v}_n .$$

Since $\vec{w}$ was an arbitrary element of W, we see that $(T\vec{v}_1, \ldots, T\vec{v}_n)$ spans W.