Math. 108A Homework 6 Due Thursday, July 31, 2008

3.3. Let $(\vec{u}_1, \ldots, \vec{u}_n)$ be a basis for $U$. By Theorem 2.12, we can extend $(\vec{u}_1, \ldots, \vec{u}_n)$ to a basis $(\vec{u}_1, \ldots, \vec{u}_n, \vec{v}_1, \ldots, \vec{v}_m)$ for $V$.

Suppose that $S\vec{u}_i = \vec{w}_i \in W$, for $1 \le i \le n$. Then define

$T : V \longrightarrow W$ by

$$T(a_1\vec{u}_1 + \cdots + a_n\vec{u}_n + b_1\vec{v}_1 + \cdots + b_m\vec{v}_m) = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n$$

If $\vec{u} \in U$, $\vec{u} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$, and

$$T(\vec{u}) = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n = a_1 S(\vec{u}_1) + \cdots + a_n S(\vec{u}_n) = S(\vec{u}).$$

3.4. Since $\vec{u} \notin \text{null}(T)$, $T(\vec{u}) = b \neq 0 \in F$

If $\vec{v} \in V$, $T(\vec{v}) = c \in F$, then

$$T(\vec{v} - \tfrac{c}{b}\vec{u}) = c - \tfrac{c}{b}\cdot b = 0, \quad \text{so } \vec{n} = \vec{v} - \tfrac{c}{b}\vec{u} \in \text{null } T$$

$\therefore \vec{v} = \vec{n} + \frac{c}{b}\vec{u}$, where $\vec{n} \in \text{null}(T)$, $\frac{c}{b}\vec{u} \in \{a\vec{u} : a \in F\}$.

Thus $V = \text{null}(T) + \{a\vec{u} : a \in F\}$.

If $\vec{v} \in null(T) \cap \{a\vec{u} : a \in F\}$, $\vec{v} = a\vec{u}$ and $T(\vec{v}) = 0$.

$\therefore$ $T(a\vec{u}) = a\,T(\vec{u}) = 0$. Since $T(\vec{u}) \neq 0$, $a = 0$.

Hence $\vec{v} = a\vec{u} = \vec{0}$, and

$null(T) \cap \{a\vec{u} : a \in F\} = \{\vec{0}\}$. $\therefore$ $V = null(T) \oplus \{a\vec{u} : a \in F\}$.

3.12. If there is a surjective linear map $T : V \to W$ then range $T = W$ and by Theorem 3.4,

$$\dim V = \dim null\, T + \dim range\, T \geq \dim W.$$

If $\dim V \geq \dim W$ we can choose bases $(\vec{v}_1, \ldots, \vec{v}_n)$ for $V$ and $(\vec{w}_1, \ldots, \vec{w}_m)$ for $W$, with $n \geq m$. Define $T : V \longrightarrow W$ by

$$T(a_1\vec{v}_1 + \cdots + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} + \cdots + a_n\vec{v}_n) = a_1\vec{w}_1 + \cdots + a_m\vec{w}_m.$$

One checks that $T$ is indeed a linear map.

If $\vec{w} \in W$, $\vec{w} = a_1\vec{w}_1 + \cdots + a_m\vec{w}_m = T(a_1\vec{v}_1 + \cdots + a_m\vec{v}_m)$, so $T$ is surjective.