Math. 108A Homework 7 Due Thursday, August 7, 2008

3.8. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for null(T). Extend to a basis $(\vec{v}_1, \ldots, \vec{v}_n, \vec{u}_1, \ldots, \vec{u}_m)$ for V. Let U be the subspace of V spanned by $(\vec{u}_1, \ldots, \vec{u}_m)$.

If $\vec{v} \in U \cap \text{null}(T)$, then

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = b_1\vec{u}_1 + \cdots + b_m\vec{u}_m .$$

Then

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n - b_1\vec{u}_1 - \cdots - b_m\vec{u}_m = \vec{0}$$

Since $(\vec{v}_1, \ldots, \vec{v}_n, \vec{u}_1, \ldots, \vec{u}_m)$ is linearly independent,

$$a_1 = \cdots = a_n = 0 = b_1 = \cdots = b_m .$$

Hence $\vec{v} = \vec{0}$. Thus $U \cap \text{null}(T) = \{\vec{0}\}$.

3.9. $\dim(\text{null}(T)) = 2$. Hence by Theorem 3.4

$$4 = \dim \mathbb{F}^4 = \dim \text{null}\, T + \dim \text{range}\, T = 2 + \dim \text{range}\, T$$

$$\Rightarrow \dim \text{range}\, T = 2.$$

$\text{range}\, T \subseteq \mathbb{F}^2$ and $\dim \text{range}\, T = 2 \Rightarrow \text{range}\, T = \mathbb{F}^2$ by Problem 2.11. Hence T is surjective.

26. Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and define $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

by $T_A(\vec{x}) = A\vec{x}$.

Then (a) holds $\iff$ T_A is injective $\iff$ null $(T_A) = \{\vec{0}\}$

while (b) holds $\iff$ T_A is surjective $\iff$ range $(T_A) = \mathbb{R}^n$.

By Problem 2.11, range $(T_A) = \mathbb{R}^n$ $\iff$ dim range $(T_A) = n$.

So we need only show that dim null $T_A = 0$ $\iff$ dim range $(T_A) = n$.

But Theorem 3.4 states that

$$n = \dim \mathbb{R}^n = \dim \operatorname{null}(T_A) + \dim \operatorname{range}(T_A)$$

so dim null $(T_A) = 0$ $\iff$ dim range $(T_A) = n$.