Math. 108A. Homework 8 Due Thursday, August 14, 2008

3.24. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $ST = TS$ for all $S \in \mathcal{L}(V) \Leftrightarrow T = aI$ for some $a \in \mathbb{F}$.

Proof $\Leftarrow$: $T = aI \Rightarrow$

$$ST = SaI = aSI = aS = aIS = TS.$$

$\Rightarrow$: We can assume $\dim V > 1$. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for V. Define $a_{ij} \in \mathbb{F}$ by

$$(T\vec{v}_1 \quad \cdots \quad T\vec{v}_n) = (\vec{v}_1 \quad \cdots \quad \vec{v}_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Strategy : Show that $a_{ij} = 0$ when $i \neq j$ and $a_{ii} = a_{jj}$ when $i \neq j$.

I. Define $S_{ij} : V \longrightarrow V$ for $i \neq j$.

$$S_{ij}(b_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = b_i\vec{v}_j$$

Then if $k \neq i$,

$$\vec{0} = TS_{ij}(\vec{v}_k) = S_{ij}T(\vec{v}_k) = S_{ij}(a_{k1}\vec{v}_1 + \cdots + a_{kn}\vec{v}_n)$$
$$= S_{ij}(a_{ki}\vec{v}_i) = a_{ki}\vec{v}_j \Rightarrow a_{ki} = 0.$$

II. $TS_{ij}(\vec{v}_i) = T(\vec{v}_j) = a_{1j}\vec{v}_1 + \cdots + a_{nj}\vec{v}_n = a_{jj}\vec{v}_j$

$S_{ij}T(v_i) = S_{ij}(a_{i1}\vec{v}_1 + \cdots + a_{in}\vec{v}_n) = a_{ii}\vec{v}_j$. Hence

$a_{ii}\vec{v}_j = a_{jj}\vec{v}_j \Rightarrow a_{ii} = a_{jj}$.

Let $a = a_{ii}$. Then $T\vec{v}_i = a\vec{v}_i$, so $T = aI$, QED

4.1. Let $p(z) = (z-1)(z-2)\cdots(z-m)$.

5.5. Suppose $T : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$ is defined by $T\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$.

Then $T\begin{pmatrix} w \\ z \end{pmatrix} = \lambda\begin{pmatrix} w \\ z \end{pmatrix} \Rightarrow \begin{cases} z = \lambda w \\ w = \lambda z \end{cases} \Rightarrow z^2 = \lambda^2 z,\ w^2 = \lambda^2 w.$

$\Rightarrow z = w = 0$ <u>or</u> $\lambda^2 = 1$. If $\begin{pmatrix} w \\ z \end{pmatrix}$ is an eigenvector

$\begin{pmatrix} w \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\lambda^2 = \pm 1$. $\therefore$ only possible eigenvalues

are $\lambda = \pm 1$. $\lambda = 1$: $\begin{matrix} z = w \\ w = z \end{matrix}$, $\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an

eigenvector. $\lambda = -1$: $\begin{matrix} z = -w \\ w = -z \end{matrix}$, $\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector.

Eigenvalues : $\lambda = \pm 1$.

$W_1 = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $W_{-1} = \text{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.