Math. 108A  Homework 9  Due Thursday, August 21, 2008

5-6. $T\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2z_2 \\ 0 \\ 5z_3 \end{pmatrix} = \lambda\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$  $\begin{cases} 2z_2 = \lambda z_1 \\ 0 = \lambda z_2 \\ 5z_3 = \lambda z_3 \end{cases}$

$5z_3 = \lambda z_3 \Rightarrow (5-\lambda)z_3 = 0 \Rightarrow \lambda = 5$ or $z_3 = 0$

If $\lambda = 5$, $0 = 5z_2 \Rightarrow z_2 = 0$, $0 = 2z_2 = 5z_1 \Rightarrow z_1 = 0$

$\lambda = 5$  $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a solution

$\lambda = 5$ is an eigenvalue with eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

If $\lambda \neq 5$, $z_3 = 0$  $0 = \lambda z_2 \Rightarrow \lambda = 0$ or $z_2 = 0$

If $\lambda \neq 0$, $z_2 = 0 = 2z_2 = \lambda z_1 \Rightarrow z_1 = 0$, so no nonzero solutions when $\lambda \neq 0, 5$

If $\lambda = 0$  $2z_2 = 0$, so $z_2 = 0$

$\lambda = 0$ : $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution

$\lambda = 0$ is an eigenvalue with eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

5-8. $T(z_1, z_2, z_3, \ldots) = (z_2, z_3, z_4, \ldots) = \lambda(z_1, z_2, z_3, \ldots)$

$\Rightarrow z_2 = \lambda z_1, \; z_3 = \lambda z_2 = \lambda^2 z_1, \; z_4 = \lambda z_3 = \lambda^3 z_1, \ldots$

$(z_1, z_2, z_3, z_4, \ldots) = z_1(1, \lambda, \lambda^2, \lambda^3, \lambda^4, \ldots)$

Any $\lambda \in \mathbb{F}$ is an eigenvalue with eigenvector $(1, \lambda, \lambda^2, \lambda^3, \ldots)$

5-12. Suppose $T$ had two distinct eigenvalues $\lambda_1$ and $\lambda_2$

Let $u_1$ and $u_2$ be corresponding eigenvectors, $Tu_1 = \lambda_1 u_1$, $Tu_2 = \lambda_2 u_2$

Theorem 5.6. $\Rightarrow$ $u_1$ and $u_2$ are linearly independent.

$T(u_1 + u_2) = T(u_1) + T(u_2) = \lambda_1 u_1 + \lambda_2 u_2$

But $u_1 + u_2$ is an eigenvector, so $T(u_1 + u_2) = \lambda(u_1 + u_2)$, for some $\lambda \in \mathbb{F}$.

But then $\lambda_1 u_1 + \lambda_2 u_2 = \lambda(u_1 + u_2) = \lambda u_1 + \lambda u_2$, and hence $(\lambda_1 - \lambda)u_1 + (\lambda_2 - \lambda)u_2 = 0$. Since $u_1$ and $u_2$ are linearly independent, $\lambda_1 - \lambda = 0 = \lambda_2 - \lambda$, so $\lambda_2 = \lambda = \lambda_1$.

Thus there is only one eigenvalue $\lambda$ and $T(u) = \lambda u$ for all $u \in V$. $\therefore T = \lambda I$.