Name: Key

Mathematics 108A: Practice Final Exam

Aug 26, 2008

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Copies of practice quizzes are available online at http://math.ucsb.edu/moore/108ASyllabus2008.htm Keys to the quizzes can be found at the same URL.

Part I. Multiple Choice. Circle the best answer to each of the following questions.

1. The vectors (1,2,1) and (2,4,t) in \mathbb{R}^3 are linearly independent if and only if

a. t=1 b. $t \neq 1$ c. t=2 (d. $t \neq 2$)

e. None of these

2. If

$$W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 + x_2 + x_3 = 0\}, \text{ and } W_2 = \text{span}(-1, 0, t),$$

then \mathbb{R}^3 is the direct sum of W_1 and W_2 if and only if

a. t = 1

b. $t \neq 1$ c. t = 2 d. $t \neq 2$

e. None of these

3 Suppose that the matrix of a linear transformation $T: V \to V$ with respect to a basis (v_1, \ldots, v_3) is

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} .$$

Then $T - \lambda I$ has nonzero null space if and only if

a. $\lambda = 1$ b. $\lambda \neq 1$ c. $\lambda = 2$ d. $\lambda \neq 2$

e. None of these

4. Suppose that V and W are finite dimensional vector spaces over a field $\mathbb F$ and that $T: V \to W$ is a linear map Then $\dim(\text{null}(T)) =$

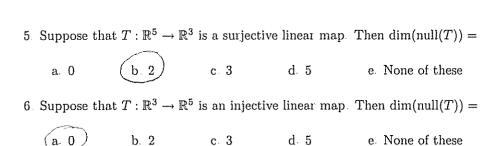
a. $\dim V - \dim(\operatorname{range}(T))$

b $\dim V + \dim(\operatorname{range}(T))$

c. $\dim W - \dim(\operatorname{range}(T))$

 $\operatorname{dim} W + \operatorname{dim}(\operatorname{range}(T))$

e None of these



7. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis for \mathbb{R}^2 defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by

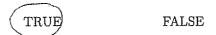
$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ x_1 + x_2 \end{pmatrix},$$

then the matrix $\mathcal{M}(T)$ of T with respect to the basis β is

a.
$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$
 b. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ d. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ e. None of these

Part II True-False. Circle the best answer

1 The space $\mathbb C$ of complex numbers can be regarded as a vector space over the field $\mathbb R$ of real numbers. When this is done, it has as one of its bases (1,i), where i is the square root of -1. Thus the dimension of $\mathbb C$ as a vector space over $\mathbb R$ is two



2. If $\beta = (\mathbf{v}_1, \dots \mathbf{v}_n)$ is a basis for a vector space V of the field \mathbb{C} of complex numbers, then $\gamma = (\mathbf{v}_1, i\mathbf{v}_1, \dots \mathbf{v}_n, i\mathbf{v}_n)$ is a basis for V when V is regarded as a vector space over the field \mathbb{R} of real numbers.



3. If W_1 and W_2 are subspaces of \mathbb{R}^n , then $W_1 \cup W_2$ is also a subspace of \mathbb{R}^n .



	te-dimensional vector space over a field $\mathbb F$, any two bases for V number of elements.
TRUE	FALSE
	be space of polynomials with coefficients in \mathbb{R} , then $\mathcal{P}(\mathbb{R})$ is an onal vector space.
TRUE	FALSE
	-dimensional vector space over \mathbb{C} , and a linear transformation n distinct eigenvalues, then there is a basis for V such that al
TRUE	FALSE
7 If $A \in M_{n \times n}$	(\mathbb{R}) then the equation
	$A\mathbf{x} = 0$ has no nonzero solutions $\mathbf{x} \in \mathbb{R}^n$
if and only if th	e equation
	$A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{R}^n$
for every choice	of \mathbf{x} in \mathbb{R}^n .
TRUE	FALSE
8. An <i>n</i> -dimensi	ional real vector space must be isomorphic to \mathbb{R}^n .
TRUE	FALSE

9. If V is a finite-dimensional real vector space, a linear transformation $T:V\to V$ always has at least one real eigenvalue.

TRUE FALSE

10. If V is a finite-dimensional complex vector space, a linear transformation $T:V\to V$ always has at least one complex eigenvalue.

TRUE) FALSE

Part III. Be able to give definitions of key terms in the course, such as:

- 1. A basis for a vector space V over the field \mathbb{F} is ...
- 2. A subspace of a vector space V over \mathbb{F} is ...
- 3. A linear transformation $T: V \to W$ is ...
- 4. If V and W are vector spaces over \mathbb{F} and $T:V\to W$ is a linear transformation, the null space null(T) of T is
- 5. If V is a vector space over $\mathbb F$ and $T:V\to V$ is a linear transformation, an eigenvalue of T is ...
- 6. Let V and W be finite-dimensional vector spaces over $F, T: V \to W$ a linear transformation, $\beta = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ a basis for $V, \gamma = (\mathbf{w}_1, \dots, \mathbf{w}_m)$ a basis for W. The matrix $\mathcal{M}(T)$ of T with respect to these bases is

Solution: the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{such that}$$

$$(T(\mathbf{v}_1) \qquad T(\mathbf{v}_n)) = (\mathbf{w}_1 \qquad \mathbf{w}_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$
Part IV. Be able to prove simple propositions such as those given in the quizze

Part IV. Be able to prove simple propositions such as those given in the quizzes and practice quizzes. For example:

- 1 Let $T:V\to W$ be a linear transformation. Prove that the null space $\operatorname{null}(T)$ of T is a subspace of V
- 2. Let $T:V\to W$ be a one-to-one linear transformation. Prove that the null space $\mathrm{null}(T)$ of T is the zero subspace of V.
- 3. Let $T: V \to W$ be a linear transformation, $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ a basis for the null space $\mathrm{null}(T)$ of T. Extend $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ to a basis $(\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_n)$ for V. Show that $(T(\mathbf{w}_{m+1}), \dots, T(\mathbf{w}_n))$ is a basis for range(T).
- 4. Let $T:V\to V$ be a linear transformation. Prove that T is one-to-one if and only if 0 is not an eigenvalue for T
- 5. Let V be a finite dimensional complex vector space, $T:V\to V$ a linear maphrove that T has at least one complex eigenvalue.
- 6 Prove the following:

Proposition. Let $\beta = (\mathbf{v}_1, \dots \mathbf{v}_n)$ be an n-tuple of distinct elements in a vector space V over a field F. If every element \mathbf{v} of V can be uniquely expressed as a linear combination of elements of β , then β is a basis for V.

Part V. Give complete answers to the following questions

1. Let $P_3(\mathbb{R})$ denote the polynomials of degree three with basis $\beta=(1,x,x^2,x^3)$. If $T:P_3(\mathbb{R})\to P_3(\mathbb{R})$ is the linear transformation defined by

$$T(f(x)) = f'(x) + 3f(x) - f(0),$$

what is the matrix $\mathcal{M}(T)$ of T with respect to β ?

2. Let $V=C^\infty(\mathbb{R}),$ the space of C^∞ functions from \mathbb{R} to itself. Define a linear transformation

$$T: V \to V$$
 by $T(f)(t) = \frac{d^2f}{dt^2}(t) + 4f(t)$

Find a basis for the null space null(T) of T.

3. Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 7 & 5 & 2 \\ 1 & 7 & 6 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Find a basis for the null space $\operatorname{null}(T)$ of T. Find a basis for the range of T.

4. What are the eigenvalues of the linear transformation $L_A: \mathbb{R}^5 \to \mathbb{R}^5$, where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}?$$

Find a basis for the space eigenspace \mathcal{W}_2 of eigenvectors corresponding to the eigenvalue 2.

Part VI. Prove the following theorem:

Theorem. If V is a vector space over F, $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ is a linearly independent list of elements of V and $V = \mathrm{span}(\mathbf{w}_1, \dots, \mathbf{w}_n)$, then $m \leq n$

Hint: Use the

Linear Dependence Lemma. Suppose that $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ is a linearly dependent list of vectors in a finite-dimensional vector space V and that $\mathbf{v}_1 \neq \mathbf{0}$. Then there exists $j \in \{2, \dots, m\}$ such that

$$\mathbf{v}_j \in span(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$$

Moreover,

$$span(\mathbf{v}_1,\ldots,\mathbf{v}_{j-1},\mathbf{v}_{j+1},\ldots,\mathbf{v}_m) = span(\mathbf{v}_1,\ldots,\mathbf{v}_m).$$