Name: Key

Mathematics 108A: Practice Final Exam

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Copies of practice quizzes are available online at
http://math.ucsb.edu/ moore/108ASyllabus2008.htm
Keys to the quizzes can be found at the same URL.

Part I. Multiple Choice. Circle the best answer to each of the following questions.

1. The vectors $(1,2,1)$ and $(2,4,t)$ in $\mathbb{R}^3$ are linearly independent if and only if

a. $t=1$ b. $t\neq 1$ c. $t=2$ d. $t\neq 2$ e. None of these

2. If

$$W_1 = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : 2x_1+x_2+x_3=0\}, \quad \text{and} \quad W_2=\text{span}(-1,0,t),$$

then $\mathbb{R}^3$ is the direct sum of W_1 and W_2 if and only if

a. $t=1$ b. $t\neq 1$ c. $t=2$ d. $t\neq 2$ e. None of these

3. Suppose that the matrix of a linear transformation $T: V\to V$ with respect to a basis $(\mathbf{v}_1,\dots,\mathbf{v}_3)$ is

$$A=\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}.$$

Then $T-\lambda I$ has nonzero null space if and only if

a. $\lambda=1$ b. $\lambda\neq 1$ c. $\lambda=2$ d. $\lambda\neq 2$ e. None of these

4. Suppose that V and W are finite dimensional vector spaces over a field $\mathbb{F}$ and that $T: V\to W$ is a linear map Then $\dim(\text{null}(T)) =$

a. $\dim V - \dim(\text{range}(T))$ b. $\dim V + \dim(\text{range}(T))$

c. $\dim W - \dim(\text{range}(T))$ d. $\dim W + \dim(\text{range}(T))$

e. None of these

5. Suppose that $T : \mathbb{R}^5 \to \mathbb{R}^3$ is a surjective linear map. Then $\dim(\text{null}(T)) =$

a. 0   b. 2   c. 3   d. 5   e. None of these

6. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^5$ is an injective linear map. Then $\dim(\text{null}(T)) =$

a. 0   b. 2   c. 3   d. 5   e. None of these

7. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis for $\mathbb{R}^2$ defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ x_1 + x_2 \end{pmatrix},$$

then the matrix $\mathcal{M}(T)$ of T with respect to the basis β is

a. $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ e. None of these

Part II True-False. Circle the best answer.

1. The space $\mathbb{C}$ of complex numbers can be regarded as a vector space over the field $\mathbb{R}$ of real numbers. When this is done, it has as one of its bases $(1, i)$, where i is the square root of -1. Thus the dimension of $\mathbb{C}$ as a vector space over $\mathbb{R}$ is two.

TRUE   FALSE

2. If $\beta = (\mathbf{v}_1, \ldots \mathbf{v}_n)$ is a basis for a vector space V of the field $\mathbb{C}$ of complex numbers, then $\gamma = (\mathbf{v}_1, i\mathbf{v}_1, \ldots \mathbf{v}_n, i\mathbf{v}_n)$ is a basis for V when V is regarded as a vector space over the field $\mathbb{R}$ of real numbers.

TRUE   FALSE

3. If W_1 and W_2 are subspaces of $\mathbb{R}^n$, then $W_1 \cup W_2$ is also a subspace of $\mathbb{R}^n$.

TRUE   FALSE

4. If V is a finite-dimensional vector space over a field $\mathbb{F}$, any two bases for V have the same number of elements.

TRUE FALSE

5. If $\mathcal{P}(\mathbb{R})$ is the space of polynomials with coefficients in $\mathbb{R}$, then $\mathcal{P}(\mathbb{R})$ is an infinite-dimensional vector space.

TRUE FALSE

6. If V is an n-dimensional vector space over $\mathbb{C}$, and a linear transformation $T : V \to V$ has n distinct eigenvalues, then there is a basis for V such that $\mathcal{M}(T)$ is diagonal.

TRUE FALSE

7. If $A \in M_{n\times n}(\mathbb{R})$ then the equation

$$A\mathbf{x} = \mathbf{0} \quad \text{has no nonzero solutions} \quad \mathbf{x} \in \mathbb{R}^n$$

if and only if the equation

$$A\mathbf{x} = \mathbf{b} \quad \text{has a unique solution} \quad \mathbf{x} \in \mathbb{R}^n$$

for every choice of $\mathbf{x}$ in $\mathbb{R}^n$.

TRUE FALSE

8. An n-dimensional real vector space must be isomorphic to $\mathbb{R}^n$.

TRUE FALSE

9. If V is a finite-dimensional real vector space, a linear transformation $T : V \to V$ always has at least one real eigenvalue.

TRUE FALSE

10. If V is a finite-dimensional complex vector space, a linear transformation $T : V \to V$ always has at least one complex eigenvalue.

TRUE FALSE

Part III. Be able to give definitions of key terms in the course, such as:

1. A *basis* for a vector space V over the field $\mathbb{F}$ is ...

2. A *subspace* of a vector space V over $\mathbb{F}$ is ...

3. A *linear transformation* $T : V \to W$ is ...

4. If V and W are vector spaces over $\mathbb{F}$ and $T : V \to W$ is a linear transformation, the *null space* null(T) of T is ...

5. If V is a vector space over $\mathbb{F}$ and $T : V \to V$ is a linear transformation, an *eigenvalue* of T is ...

6. Let V and W be finite-dimensional vector spaces over F, $T : V \to W$ a linear transformation, $\beta = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ a basis for V, $\gamma = (\mathbf{w}_1, \dots, \mathbf{w}_m)$ a basis for W. The *matrix* $\mathcal{M}(T)$ of T with respect to these bases is ...

Solution: the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{such that}$$

$$\begin{pmatrix} T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Part IV. Be able to prove simple propositions such as those given in the quizzes and practice quizzes. For example:

1. Let $T : V \to W$ be a linear transformation. Prove that the null space null(T) of T is a subspace of V.

2. Let $T : V \to W$ be a one-to-one linear transformation. Prove that the null space null(T) of T is the zero subspace of V.

3. Let $T : V \to W$ be a linear transformation, $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ a basis for the null space null(T) of T. Extend $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ to a basis $(\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_n)$ for V. Show that $(T(\mathbf{w}_{m+1}), \dots, T(\mathbf{w}_n))$ is a basis for range(T).

4. Let $T : V \to V$ be a linear transformation. Prove that T is one-to-one if and only if 0 is not an eigenvalue for T.

5. Let V be a finite dimensional complex vector space, $T : V \to V$ a linear map. Prove that T has at least one complex eigenvalue.

6. Prove the following:

Proposition. *Let $\beta = (\mathbf{v}_1, \dots \mathbf{v}_n)$ be an n-tuple of distinct elements in a vector space V over a field F. If every element $\mathbf{v}$ of V can be uniquely expressed as a linear combination of elements of β, then β is a basis for V.*

Part V. Give complete answers to the following questions.

1. Let $P_3(\mathbb{R})$ denote the polynomials of degree three with basis $\beta = (1, x, x^2, x^3)$. If $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is the linear transformation defined by

$$T(f(x)) = f'(x) + 3f(x) - f(0),$$

what is the matrix $\mathcal{M}(T)$ of T with respect to β?

2. Let $V = C^\infty(\mathbb{R})$, the space of C^∞ functions from $\mathbb{R}$ to itself. Define a linear transformation

$$T : V \to V \quad \text{by} \quad T(f)(t) = \frac{d^2 f}{dt^2}(t) + 4f(t).$$

Find a basis for the null space null(T) of T.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 7 & 5 & 2 \\ 1 & 7 & 6 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Find a basis for the null space null(T) of T. Find a basis for the range of T.

4. What are the eigenvalues of the linear transformation $L_A : \mathbb{R}^5 \to \mathbb{R}^5$, where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}?$$

Find a basis for the space eigenspace W_2 of eigenvectors corresponding to the eigenvalue 2.

Part VI. Prove the following theorem:

Theorem. *If V is a vector space over F, $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list of elements of V and $V = span(\mathbf{w}_1, \ldots, \mathbf{w}_n)$, then $m \leq n$.*

Hint: Use the

Linear Dependence Lemma. *Suppose that $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a linearly dependent list of vectors in a finite-dimensional vector space V and that $\mathbf{v}_1 \neq \mathbf{0}$. Then there exists $j \in \{2, \ldots, m\}$ such that*

$$\mathbf{v}_j \in span(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$$

Moreover,

$$span(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = span(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$