Name: Key

Mathematics 108A: Quiz 2
July 15, 2008
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Part I. True-False. Circle the best answer to each of the following questions. Each question is worth 2 points.

1. If

$$W_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 0\}, \quad \text{and} \quad W_2 = \text{span}(2, -2),$$

then $\mathbb{R}^2$ is the direct sum of W_1 and W_2.

TRUE FALSE

2. If U_1, U_2 and W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W$$

then $U_1 = U_2$.

TRUE FALSE

3. If $V = \{$ functions $f : \mathbb{R} \to \mathbb{R}$ $\}$, a vector space over $\mathbb{R}$ with addition and scalar multiplication defined by

$$(f+g)(t) = f(t) + g(t), \quad (af)(t) = a(f(t)), \quad \text{for } f, g \in V \text{ and } a \in \mathbb{R},$$

then the list (f, g), where

$$f(t) = \sin t, \qquad g(t) = \cos t,$$

is linearly independent.

TRUE FALSE

4. If V is the same vector space considered in the previous problem, then the list (f, g), where

$$f(t) = e^t, \qquad g(t) = 3e^t,$$

is linearly independent.

TRUE FALSE

5. Let V be the subspace of $\mathbb{R}^4$ defined by

$$V = \operatorname{span}((1,2,0,5),(0,0,1,4),(2,4,0,10))$$

Then $((1,2,0,5),(0,0,1,4))$ is a basis for V.

TRUE FALSE

6. If V is a finite-dimensional vector space, the length of every spanning list of vectors is less than or equal to the length of every linearly independent list.

TRUE FALSE

7. A vector space V is infinite-dimensional if and only if V contains an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of vectors such that $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is linearly independent for every positive integer n.

TRUE FALSE

8. Let $\mathcal{P}(F)$ denote the space of polynomials

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

of arbitrary degree, the coefficients a_i being in F. Then $\mathcal{P}(F)$ is infinite-dimensional.

TRUE FALSE

Part II. Give complete proofs of each of the following statements.

1. (8 points) Suppose that $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a linearly dependent list of vectors in V and that $\mathbf{v}_1 \neq \mathbf{0}$. Prove that there exists $j \in \{2, \ldots, m\}$ such that

$$\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}).$$

$\vec{v}_1, \ldots, \vec{v}_m$ linearly dependent $\Rightarrow a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0}$

for some $a_1, \ldots, a_m \in F$ not all zero.

Let j be the largest integer such that $a_j \neq 0$.

$j \geq 2$ because $\vec{v}_1 \neq \vec{0}$. $a_j\vec{v}_j = -a_1\vec{v}_1 - \cdots - a_{j-1}\vec{v}_{j-1}$

$\vec{v}_j = -\frac{a_1}{a_j}\vec{v}_1 - \cdots - \frac{a_{j-1}}{a_j}\vec{v}_{j-1}$ $\therefore$ $\vec{v}_j \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$

2. (7 points) With the same conditions as in the previous problem, prove that

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

Clearly $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m) \subseteq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$

Need to show $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m) \subseteq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m)$

If $\vec{v} \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$,

$\vec{v} = a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} + a_j\vec{v}_j + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m.$

But $\vec{v}_j \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$, so $\vec{v}_j = b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1}$.

Hence

$$\begin{aligned}\vec{v} &= a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} + a_j(b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1}) \\ &\qquad + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m \\ &= (a_1 + a_jb_1)\vec{v}_1 + \cdots + (a_{j-1} + a_jb_{j-1})\vec{v}_{j-1} \\ &\qquad + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m,\end{aligned}$$

so $\vec{v} \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m)$.