Name: Key

Mathematics 108A: Quiz 3
July 24, 2008
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Part I. True-False. Circle the best answer to each of the following questions. Each question is worth 2 points.

1. The function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is a linear map.

TRUE FALSE

2. Let $\mathbb{R}^\infty$ be the set of infinite sequences $(x_1, x_2, \ldots, x_i, \ldots)$, where each x_i is a real number. The function $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$T(x_1, x_2, x_3, \ldots) = (1, x_1, x_2, x_3, \ldots)$$

is a linear map.

TRUE FALSE

3. Let $C^0(\mathbb{R}) = \{$ continuous functions $f : \mathbb{R} \to \mathbb{R}$ $\}$, a vector space over $\mathbb{R}$ with addition and scalar multiplication defined by

$$(f+g)(t) = f(t) + g(t), \quad (af)(t) = a(f(t)), \quad \text{for } f, g \in V \text{ and } a \in \mathbb{R}.$$

Then

$$T : C^0(\mathbb{R}) \to \mathbb{R}, \quad \text{defined by} \quad T(f) = 5\int_0^1 f(t)dt,$$

is a linear map.

TRUE FALSE

4. Let A be an $m \times n$ matrix with real coefficients. The range of the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by $T_A(\mathrm{x}) = A\mathrm{x}$, is the space spanned by the rows of A.

TRUE FALSE

5. Let $V = \{$ infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ $\}$, a vector space over $\mathbb{R}$, and if t denotes the variable in the range, let

$$T : V \longrightarrow V \quad \text{be the linear map defined by} \quad T(f) = \frac{df}{dt} + 5f.$$

Then the null space of T is the space of solutions to the differential equation

$$\frac{df}{dt} + 5f = 0.$$

TRUE FALSE

Part II. Give complete answers to each of the following questions.

1. (7 points) Find a basis for the null space of the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$, defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$x_1 - x_2 - 2x_4 - 3x_5 = 0$$
$$x_3 - 5x_4 - 4 \cdot x_5 = 0$$

$$\begin{cases} x_1 = x_2 + 2x_4 + 3x_5 \\ x_2 = x_2 \\ x_3 = 5x_4 + 4x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}$

2. (7 points) Suppose that $T : V \to W$ is a linear transformation. Show that

$$\text{Null}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

is a (linear) subspace of V.

1. $T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$

$\Rightarrow \vec{0} = T(\vec{0}) \Rightarrow \vec{0} \in \text{Null}(T)$

2. If $\vec{v}, \vec{w} \in \text{Null}(T)$, $T(\vec{v}) = \vec{0}$, $T(\vec{w}) = \vec{0}$.

Hence $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$

$\therefore \vec{v} + \vec{w} \in \text{Null}(T)$.

3. If $\vec{v} \in \text{Null}(T)$, $a \in F$, then $T(\vec{v}) = \vec{0}$.

Hence $T(a\vec{v}) = a\,T(\vec{v}) = a \cdot \vec{0} = \vec{0}$

$\therefore a\vec{v} \in \text{Null}(T)$.

3. (7 points) Recall the basic lemma from Chapter 2 of the text by Axler:

Linear Dependence Lemma. *Suppose that* $(v_1, \ldots, v_m)$ *is a linearly dependent list of vectors in a finite-dimensional vector space* V *and that* $v_1 \neq 0$. *Then there exists* $j \in \{2, \ldots, m\}$ *such that* $v_j \in span(v_1, \ldots, v_{j-1})$. *Moreover,*

$$span(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_m) = span(v_1, \ldots, v_m).$$

Use the Linear Dependence Lemma to prove the following:

Theorem. *Every spanning list in a vector space* V *can be reduced to a basis.*

Hint: The idea of the proof is to start with a spanning list and throw away elements until you have a basis

Suppose that $V = \text{span}(\vec{v}_1, \ldots, \vec{v}_n)$

Start with the list $L_0 = (\vec{v}_1, \ldots, \vec{v}_n)$

For each j, $1 \leq j \leq n$, ask whether $\vec{v}_j$ is in the span of the list L_{j-1}. If so, throw it away obtaining a new list L_j. If not, let $L_j = L_{j-1}$

Repeat this process n times obtaining a list L_n which spans V.

If the list L_n were linearly dependent, one of the $\vec{v}_j$'s in L_n would be in the span of the previous elements by the Linear Dependence Lemma.

This is not the case, so L_n is linearly independent

$\therefore$ L_n is a basis for V.