Name: Key

Mathematics 108A: Quiz 4

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**Part I. True-False.** Circle the best answer to each of the following questions Each question is worth 2 points.

1 Let $\beta = \{u_1, \dots u_n\}$ be a finite subset of a vector space $V$ over a field $F$. Then $\beta$ is a basis for $V$ if and only if every element $v$ of $V$ can be uniquely expressed as a linear combination of elements of $\beta$

TRUE FALSE

2. Let $\mathbb{R}^\infty$ be the set of infinite sequences $(x_1, x_2, \dots, x_i, \dots)$, where each $x_i$ is a real number The linear map $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

has a nonzero null space spanned by the vector $(1, 0, 0, \dots)$.

TRUE FALSE

3. matrix $A$ defines a linear map

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{by} \quad T_A(\mathrm{x}) = A\mathrm{x}.$$

The null space of this linear map is the space of solutions to the homogeneous linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots \quad &\quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned} \tag{1}$$

TRUE FALSE

4. The range of the linear map $T_A$ is the space of vectors $\mathrm{b} = (b_1, \dots, b_m)$ in $\mathbb{R}^m$ such that the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \cdots \quad &\quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

has a solution.

TRUE FALSE

5. Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation and

$$\text{null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 - 2x_4 = 0, x_3 + x_4 = 0\}.$$

Then $T$ must be surjective.

TRUE FALSE

**Part II.** Give complete answers to each of the following questions.

1. (7 points) Let $\mathcal{P}_2(\mathbb{R})$ denote the space of polynomials of degree two, with basis $\beta = (p_0, p_1, p_2)$, where

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2.$$

Suppose that $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is the linear transformation defined by

$$T(p(x)) = \frac{dp}{dx}(x) - 7p(x).$$

What is the matrix $\mathcal{M}(T, \beta, \beta)$ of $T$ with respect to this basis?

$$T(1) = -7 = (1 \quad x \quad x^2)\begin{pmatrix} -7 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x) = 1 - 7x = (1 \quad x \quad x^2)\begin{pmatrix} 1 \\ -7 \\ 0 \end{pmatrix}$$

$$T(x^2) = 2x - 7x^2 = (1 \quad x \quad x^2)\begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}$$

$$\mathcal{M}(T, \beta, \beta) = \begin{pmatrix} -7 & 1 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -7 \end{pmatrix}$$

2. (7 points) a. Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in $V$ is linearly independent if and only if ..

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0} \quad \Rightarrow \quad a_1 = a_2 = \cdots = a_n = 0$$

b. Suppose that $T : V \to W$ is an injective linear map, and that $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a linearly independent list of vectors in $V$. Prove that $(T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n))$ is a linearly independent list of vectors in $W$.

Hint: Start by assuming

$$a_1 T(\mathbf{v}_1) + \cdots + a_n T(\mathbf{v}_n) = 0$$

Then $T(a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n) = \vec{0}$

Hence $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n \in \text{null}(T)$

But $\text{null}(T) = \{\vec{0}\}$, so $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$

Since $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent

$$a_1 = a_2 = \cdots = a_n = 0.$$

It follows therefore that $(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ is linearly independent.

Hint: Finish by showing that

$$a_1 = a_2 = \cdots = a_n = 0.$$

3. The Main Theorem from Chapter 3 of the text by Axler is:

**Theorem.** *If $V$ is a finite dimensional vector space and $T : V \to W$ is a linear map into a vector space $W$, then*

$$\dim V = \dim \mathit{null}(T) + \dim \mathit{range}(T).$$

Recall the idea behind the proof. We start by choosing a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ for null($T$). The Extension Theorem from Chapter 2 states that we can extend this to a basis

$$(\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n)$$

of $V$. If we can show that $(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n))$ is a basis for range($T$), then $\dim \mathrm{range}(T) = n$. It will then follow that

$$\dim V = m + n = \dim \mathrm{null}(T) + \dim \mathrm{range}(T),$$

and the theorem will be proven. Thus we need only show that $(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n))$ is linearly independent and spans range($T$).

Prove that the list $(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n))$ spans range($T$).

Suppose $\vec{w} \in$ range$(T)$. Then $\vec{w} = T(\vec{v})$ where $\vec{v} \in V$. We can write

$$\vec{v} = a_1 \vec{u}_1 + \cdots + a_m \vec{u}_m + b_1 \vec{v}_1 + \cdots + b_n \vec{v}_n$$

Then

$$\vec{w} = T(\vec{v}) = T(a_1 \vec{u}_1 + \cdots + a_m \vec{u}_m + b_1 \vec{v}_1 + \cdots + b_n \vec{v}_n)$$

$$= a_1 T(\vec{u}_1) + \cdots + a_m T(\vec{u}_m) + b_1 T(\vec{v}_1) + \cdots + b_n T(\vec{v}_n)$$

Since $\vec{u}_1, \ldots, \vec{u}_m \in$ null$(T)$,

$$\vec{w} = b_1 T(\vec{v}_1) + \cdots + b_n T(\vec{v}_n).$$

$\therefore$ $(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ spans range$(T)$.