Name: Key

Mathematics 108A: Quiz 5

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I. True-False. Circle the best answer to each of the following questions.

1. Suppose that V is a complex vector space of dimension n and that $T : V \to V$ is a linear map. If $\mathbf{v}$ is a nonzero element of V, then

$$(\mathbf{v}, T(\mathbf{v}), \ldots, T^n(\mathbf{v}))$$

is linearly dependent. Hence there are complex constants a_0, a_1, $\ldots$, a_n such that

$$a_0\mathbf{v} + a_1 T(\mathbf{v}) + \cdots + a_n T^n(\mathbf{v}) = \mathbf{0}. \tag{1}$$

TRUE FALSE

2. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial with complex coefficients, then it follows from the fundamental theorem of algebra that

$$p(z) = c(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n),$$

for some choice of complex numbers c and $\lambda_1, \ldots \lambda_n$.

TRUE FALSE

3. Suppose that V is a complex vector space of dimension n and that $T : V \to V$ is a linear map. A complex number λ is an eigenvalue for T if and only if the linear map $T - \lambda I$ is not injective.

TRUE FALSE

4. Let $\mathbb{C}^\infty$ denote the space of infinite sequences $(z_1, z_2, z_3, \ldots)$, where each z_i is a complex number and that $T : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is the linear map such that

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots).$$

Then T must have at least one complex eigenvalue.

TRUE FALSE

5. If V is a vector space over $\mathbb{F}$, $T : V \to V$ is a linear transformation and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, then there is a smallest integer $k \in \{2, 3, \ldots, m\}$ such that

$$\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}).$$

Hence $\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}$, for some choice of $a_1, \ldots, a_{k-1}$ in $\mathbb{F}$.

TRUE FALSE

6. Suppose that $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by

$$T_A\mathbf{x} = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$. Then T_A must have at least one real eigenvalue.

TRUE FALSE

II. Complete answers. Write out complete solutions to each of the following problems.

1. a. (2 points) Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V is linearly independent if and only if ...

$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$

$\Rightarrow a_1 = 0, \ldots, a_n = 0.$

b. (2 points) Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V spans a vector space V if and only if ...

$\vec{v} \in V \Rightarrow \vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$

for some $a_0, \ldots, a_n \in \mathbb{F}$.

c. (2 points) Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V is a basis for V if and only if ...

it is linearly independent and spans V.

2. (7 points) Prove that if V and W are two vector spaces of the same dimension, they are isomorphic. (Hint: Start by choosing bases for V and W.)

Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for V,

$(\vec{w}_1, \ldots, \vec{w}_n)$ a basis for W.

Define $T : V \longrightarrow W$ by

$$T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n.$$

Since $(\vec{w}_1, \ldots, \vec{w}_n)$ spans W,

$\vec{w} \in W \Rightarrow \vec{w} = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n = T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)$

so T is surjective.

If $T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = \vec{0}$, then

$a_1\vec{w}_1 + \cdots + a_n\vec{w}_n = \vec{0}$.

Since $(\vec{w}_1, \ldots, \vec{w}_n)$ is liniarly independent $a_1 = 0, \ldots, a_n = 0$

$\therefore a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$

Hence $T(\vec{v}) = \vec{0} \Rightarrow \vec{v} = \vec{0} = null(T) = 0$

$\therefore T$ is injective.

Since T is both injective and surjective,

T is an isomorphism.

3. a. (4 points) Suppose that $T_A : \mathbb{R}^5 \to \mathbb{R}^5$ is the linear map defined by

$$T_A \mathbf{x} = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & -3 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & -2 & -4 \end{pmatrix}.$$

Find a basis for the null space of T_A.

$$\text{null}(T_A) = \left\{ \vec{x} \in \mathbb{R}^5 : \begin{pmatrix} 1 & -3 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & -2 & -4 \end{pmatrix} \vec{x} = \vec{0} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5 : \begin{pmatrix} 1 & -3 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \vec{0} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5 : \begin{array}{l} x_1 - 3x_2 - x_4 - 2x_5 = 0 \\ x_3 - x_4 - 2x_5 = 0 \end{array} \right\}$$

$$\begin{aligned} x_1 &= 3x_2 + x_4 + 2x_5 \\ x_2 &= x_2 \\ x_3 &= x_4 + 2x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Basis for null}(T_A) = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

b. (2 points) Find the dimension of the range of T_A.

$$\dim \text{null}(T_A) = 3$$

$$\begin{aligned} 5 = \dim \mathbb{R}^5 &= \dim \text{null}(T_A) + \dim \text{range}(T_A) \\ &= 3 + \dim \text{range}(T_A) \end{aligned}$$

$$\dim \text{range}(T_A) = 2.$$