Name: Key

Mathematics 108A: Practice Quiz B

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Part I. True-False. Circle the best answer to each of the following questions. Each question is worth 2 points.

1. If

$$W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 = 0\}, \quad \text{and} \quad W_2 = \operatorname{span}(1, 2, 3),$$

then $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

TRUE FALSE

2. The list of vectors $((1, 3, 2), (2, 6, 4))$ is linearly independent in $\mathbb{R}^3$.

TRUE FALSE

3. Let V be the subspace of $\mathbb{R}^4$ defined by

$$V = \operatorname{span}((1, 2, 0, 5), (0, 0, 1, 4))$$

Then $((1, 2, 0, 5), (0, 0, 1, 4))$ is a basis for V

TRUE FALSE

4. When $\mathbb{C}^2$ is regarded as a vector space over $\mathbb{R}$, it has the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, where

$$\mathbf{v}_1 = (1, 0), \quad \mathbf{v}_2 = (i, 0), \quad \mathbf{v}_3 = (0, 1), \quad \mathbf{v}_4 = (0, i).$$

TRUE FALSE

5. Let $\mathcal{P}_m(F)$ denote the space of polynomials $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ with degree at most m, the coefficients a_i being in F. Then $\mathcal{P}_m(F)$ is a finite-dimensional vector space over F with basis $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$, where

$$\mathbf{v}_0 = 1, \quad \mathbf{v}_1 = z, \quad \ldots, \quad \mathbf{v}_m = z^m.$$

TRUE FALSE

6. Let $M_{m,n}(\mathbb{C})$ denote the space of $m \times n$ matrices over the complexes. If $V = M_{2,2}(\mathbb{C})$ is regarded as a vector space over $\mathbb{R}$, then V has dimension eight.

TRUE FALSE

7. Regard $V = M_{2,2}(\mathbb{C})$ as a vector space over $\mathbb{R}$, and let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then the list (U, I, J, K) is a basis for the subspace

$$\mathbb{H} = \{A \in M_{2,2}(\mathbb{C}) : A = tU + xI + yJ + zK, \text{ where } t, x, y, z \in \mathbb{R}\}.$$

TRUE FALSE

8. If addition is vector addition and multiplication is matrix multiplication, then $\mathbb{H}$ satisfies all of the field axioms except for commutativity of multiplication.

TRUE FALSE

Part II. Give complete answers to each of the following problems.

1. Let V be the subspace of $\mathbb{R}^5$ consisting of the vectors $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ such that

$$\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 2 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \mathbf{0}.$$

Find a basis for V

$$\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 2 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ x_5 = 0 \end{cases} \qquad \begin{cases} x_1 = -x_2 + 2x_3 - 3x_4 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = 0 \end{cases}$$

$$\vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Basis: $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Part III. Give complete proofs of each of the following statements.

1. Suppose that $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a linearly dependent list of vectors in V and that $\mathbf{v}_1 \neq \mathbf{0}$. Prove that there exists $j \in \{2, \ldots, m\}$ such that

$$\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}).$$

$\vec{v}_1, \ldots, \vec{v}_m$ linearly dependent $\Rightarrow a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = 0$ for some $a_1, \ldots, a_m \in F$ not all zero.

Let j be the largest integer such that $a_j \neq 0$.

$j \geq 2$ because $\vec{v}_1 \neq \vec{0}$. $\quad a_j\vec{v}_j = -a_1\vec{v}_1 - \cdots - a_{j-1}\vec{v}_{j-1}$

$\vec{v}_j = -\frac{a_1}{a_j}\vec{v}_1 - \cdots - \frac{a_{j-1}}{a_j}\vec{v}_{j-1}$. $\quad \therefore \vec{v}_j \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$.

2. With the same conditions as in the previous problem, prove that

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

Clearly $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m) \subseteq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$

Need to show $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m) \subseteq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m)$

If $\vec{v} \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$,

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} + a_j\vec{v}_j + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m.$$

But $\vec{v}_j \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$, so $\vec{v}_j = b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1}$. Hence

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} + a_j(b_1\vec{v}_1 + \cdots + b_{j-1}\vec{v}_{j-1}) + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m$$

$$= (a_1 + a_j b_1)\vec{v}_1 + \cdots + (a_{j-1} + a_j b_{j-1})\vec{v}_{j-1} + a_{j+1}\vec{v}_{j+1} + \cdots + a_m\vec{v}_m,$$

so $\vec{v} \in \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_m)$.

3. Suppose that you know the following lemma:

Linear Dependence Lemma. *Suppose that* $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ *is a linearly dependent list of vectors in a finite-dimensional vector space* V *and that* $\mathbf{v}_1 \neq \mathbf{0}$. *Then there exists* $j \in \{2, \dots, m\}$ *such that*

$$\mathbf{v}_j \in \mathit{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}).$$

Moreover,

$$\mathit{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots \mathbf{v}_m) = \mathit{span}(\mathbf{v}_1, \dots, \mathbf{v}_m).$$

Assuming this lemma, prove the following theorem:

Theorem. *If* V *is a finite-dimensional vector space over* F, $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ *is a linearly independent list of elements of* V *and* $V = \mathit{span}(\mathbf{w}_1, \dots, \mathbf{w}_n)$, *then* $m \leq n$.

First we note that $\vec{u}_1 \in \text{span}(\vec{w}_1, \dots, \vec{w}_n)$ so $(\vec{u}_1, \vec{w}_1, \dots, \vec{w}_n)$ is linearly dependent. Moreover $\vec{u}_1 \neq \vec{0}$ because $(\vec{u}_1, \dots, \vec{u}_m)$ is linearly independent. By Linear Dependence Lemma, $V = \text{span}(u_1, \vec{w}_1, \dots, \vec{w}_{j-1}, \vec{w}_{j+1}, \dots, \vec{w}_n)$ for some j.

Suppose now that we have shown that

$$V = \text{span}(\vec{u}_1, \dots, \vec{u}_{j-1}, \vec{w}_{\alpha(j)}, \dots, \vec{w}_{\alpha(n)})$$

Then $(\vec{u}_1, \dots, \vec{u}_{j-1}, \vec{u}_j, \vec{w}_{\alpha(j)}, \dots, \vec{w}_{\alpha(n)})$ is linearly dependent. Thus one of the elements in this list is a linear combination of its predecessors by the Linear Dependence Lemma. This element cannot be one of the $\vec{u}$'s because $(\vec{u}_1, \dots, \vec{u}_m)$ is linearly independent. Thus we can eliminate one of the $\vec{w}$'s and $V = \text{span}(\vec{u}_1, \dots, \vec{u}_j, \vec{w}_{\beta(j+1)}, \dots, \vec{w}_{\beta(n)})$

We can carry out this procedure for $j = 2, \dots, m$. After step m, $V = \text{span}(\vec{u}_1, \dots, \vec{u}_m, \vec{w}_{\alpha(m+1)}, \dots, \vec{w}_{\alpha(n)})$ $\therefore m \leq n$.