Mathematics 108A: Practice Quiz C

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Recall the basic lemma from Chapter 2 of the text by Axler:

Linear Dependence Lemma. *Suppose that* $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ *is a linearly dependent list of vectors in a finite-dimensional vector space* V *and that* $\mathbf{v}_1 \neq \mathbf{0}$. *Then there exists* $j \in \{2, \dots, m\}$ *such that* $\mathbf{v}_j \in span(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$. *Moreover,*

$$span(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots \mathbf{v}_m) = span(\mathbf{v}_1, \dots, \mathbf{v}_m).$$

Assuming this lemma, we then prove the MAIN RESULT of Chapter 2:

Comparison Theorem. *If* V *is a finite-dimensional vector space over* F, $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ *is a linearly independent list of elements of* V *and* V *is the span of a list* $(\mathbf{w}_1, \dots, \mathbf{w}_n)$, *then* $m \leq n$.

The Linear Dependence Lemma can be used to prove the most difficult of the other theorems in Chapter 2. For example:

1. Use the Linear Dependence Lemma to prove the following:

Theorem. *Every subspace* U *of a finite-dimensional vector space is finite-dimensional.*

Idea: The idea of the proof is to generate a list of vectors $(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$ in U which are linearly independent. Then show that either $U = \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$ or one can extend to a linearly independent list $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ by Linear Dependence Lemma. We must get a spanning list before the length of the list exceeds the dimension of V.

Proof: Since V is finite-dimensional, $V = \text{span}(\mathbf{w}_1, \dots, \mathbf{w}_n)$, where n is the dimension of V.

Step 1. If $U = \{\mathbf{0}\}$, then U is finite-dimensional. If $U \neq \{\mathbf{0}\}$, choose a nonzero $\underline{v}_1 \in U$. Then $(\mathbf{v}_1)$ is linearly independent.

Step k. Suppose that we have constructed a linearly independent list $(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$ within U. If $U = \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$, U is finite-dimensional. Otherwise, by the Linear Dependence Lemma, we can extend to a linearly independent list $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ within U.

The process must terminate at some step $k \leq n$ by the Comparison Theorem.

2. Use the Linear Dependence Lemma to prove the following:

Theorem. *Every spanning list in a vector space V can be reduced to a basis.*

Idea: The idea of the proof is to start with a spanning list and throw away elements until you have a basis. As long as you don't have a basis, the Linear Dependence Lemma says that you can throw something away.

Proof: Suppose that $V = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Start with the list $L = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

For each j, $1 \leq j \leq n$ ask whether $\mathbf{v}_j$ is in the span of the previous elements of the list. If so, throw it away, obtaining a new list. If not, keep $\mathbf{v}_j$ in the list.

Repeat this procedure n times obtaining a new list L which spans V. The Linear Dependence Lemma implies that this list is linearly independent. Hence L is a basis for V.

3. Use the Linear Dependence Lemma to prove the following:

Theorem. *Every linearly independent list in a finite-dimensional vector space V can be extended to a basis*

Idea: The idea of the proof is to suppose that $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. One by one, add the $\mathbf{w}_i$'s to the list L, throwing away any additions that make the list linearly dependent (using the Linear Dependence Lemma).

Proof: Suppose that $L = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$.

Step 1. If $\mathbf{w}_1$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_1$ to the end of the list. $\mathbf{w}_1$ is in the span of the resulting list and the Linear Dependence Lemma shows that the list is linearly independent.

Step k. Suppose that we have constructed a linearly independent list L extending $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ such that $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$ are in the span of L. If $\mathbf{w}_k$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_k$ to the list. $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are in the span of the new list and the list is linearly independent by the Linear Dependence Lemma.

After n steps, one obtains a list which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$, spans W and is linear independent. The resulting list is a basis which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$.