Name: Key

Mathematics 108A: Practice Quiz D

July 22, 2008

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I. True-False. Circle the best answer to each of the following questions.

1 The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 2x_1 + x_2 \\ 3x_1 + x_2 \end{pmatrix}$$

is a linear map.

TRUE FALSE

2. The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 \\ x_2^2 \end{pmatrix}$$

is a linear map.

TRUE FALSE

3. The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is a linear map.

TRUE FALSE

4. The function $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$T(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} 1 + i & 4 \\ 2 - i & 3 + 2i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

is a linear map.

TRUE FALSE

5. The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is a linear map.

TRUE FALSE

6. Let $\mathbb{R}^\infty$ be the set of infinite sequences $(x_1, x_2, \ldots, x_i, \ldots)$, where each x_i is a real number. The function $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, 0, \ldots)$$

is a linear map.

TRUE FALSE

7. Let $V = \{$ continuous functions $f : \mathbb{R} \to \mathbb{R}$ $\}$, a vector space over $\mathbb{R}$ with addition and scalar multiplication defined by

$$(f+g)(t) = f(t) + g(t), \quad (af)(t) = a(f(t)), \quad \text{for } f, g \in V \text{ and } a \in \mathbb{R}.$$

Then

$$T : V \to \mathbb{R}, \quad \text{defined by} \quad T(f) = \int_0^1 f(t)dt + 3,$$

is a linear map.

TRUE FALSE

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \frac{y^3 - 4x^2 y}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

Then $f(x,y) = 0$ when $y = \pm 2x$ or $y = 0$, but $f(x,y) = y$, when $x = 0$. (This function might help you with problem 2 on the homework assignment.)

TRUE FALSE

9. The null space of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

is the straight line through the origin spanned by the vector $(2, -1)$.

TRUE FALSE

10. The null space of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

has a basis $((-2, -3, 1))$.

TRUE FALSE

11. The range of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

has a basis $((1, 0, 0), (0, 1, 0))$.

TRUE FALSE

12. Let $\mathcal{P}(\mathbb{R})$ denote the space of polynomials with coefficients in $\mathbb{R}$. If $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is the linear map defined by $T(p) = p'$, where p' is the derivative of p, then the null space of T is spanned by the constant polynomials.

TRUE FALSE

13. Let $V = \{$ infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ $\}$, a vector space over $\mathbb{R}$, and if t denotes the variable in the range, let

$$T : V \longrightarrow V \quad \text{by} \quad T(f) = \frac{d^2 f}{dt^2} + f.$$

Then the null space of T is the space of solutions to the differential equation

$$\frac{d^2 f}{dt^2} + f = 0.$$

TRUE FALSE

Part II. Give complete answers to each of the following questions.

1. Find a basis for the null space of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 + x_4 \\ x_2 &= 3x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \qquad \vec{x} = x_3 \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Basis for null(T) : $\{(2,3,1,0), (1,1,0,1)\}$

2. Suppose that $T : V \to W$ is a linear transformation. Show that

$$\text{Range}(T) = \{\mathrm{w} \in W : T(\mathrm{v}) = \mathrm{w} \text{ for some } \mathrm{v} \in V\}$$

is a (linear) subspace of V.

$T(\vec{0}) = \vec{0}$ so $\vec{0} \in$ Range(T).

$\vec{w}_1, \vec{w}_2 \in$ Range(T) $\Rightarrow$ $\exists\, \vec{v}_1, \vec{v}_2 \in V$ with $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$.

Hence $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{w}_1 + w_2$. $\therefore$ $\vec{w}_1 + \vec{w}_2 \in$ Range(T)

$a \in F$, $\vec{w} \in$ Range(T) $\Rightarrow$ $\exists\, \vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.

Hence $T(a\vec{v}) = aT(\vec{v}) = a\vec{w}$. $\therefore$ $a\vec{w} \in$ Range(T).