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Mathematics 108A: Practice Quiz E
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Professor J. Douglas Moore

The Main Theorem from Chapter 3 of the text by Axler is:

Theorem. *If V is a finite dimensional vector space and $T : V \to W$ is a linear map into a vector space W, then*

$$\dim V = \dim \mathit{null}(T) + \dim \mathit{range}(T).$$

The proof of this theorem is based upon the theory of linear independence and bases presented in Chapter 2. We start by choosing a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ for null(T). The Extension Theorem from Chapter 2 states that we can extend this to a basis

$$(\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n)$$

of V. If we can show that $(T(\mathbf{w}_1), \ldots, T(\mathbf{w}_n))$ is a basis for range(T), then $\dim \mathrm{range}(T) = n$. It will then follow that

$$\dim V = m + n = \dim \mathrm{null}(T) + \dim \mathrm{range}(T),$$

and the theorem will be proven. Thus we need only carry out the following two steps.

1. Prove that the list $(T(\mathbf{w}_1), \ldots, T(\mathbf{w}_n))$ spans range(T).

Solution: Suppose that $\mathbf{w} \in \mathrm{range}(T)$. Then there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. We can write

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m + b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n,$$

and then

$$T(\mathbf{v}) = a_1 t(\mathbf{u}_1) + \cdots + a_m T(\mathbf{u}_m) + b_1 T(\mathbf{w}_1) + \cdots + b_n T(\mathbf{w}_n),$$

and since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ lie in null(T),

$$\mathbf{w} = T(\mathbf{v}) = b_1 T(\mathbf{w}_1) + \cdots + b_n T(\mathbf{w}_n).$$

Thus $\mathbf{w}$ lies in the span of $(T(\mathbf{w}_1), \ldots, T(\mathbf{w}_n))$.

2. Prove that the list $(T(\mathbf{w}_1), \ldots, T(\mathbf{w}_n))$ is linearly independent.

Solution: Suppose that

$$b_1 T(\mathbf{w}_1) + \cdots + b_n T(\mathbf{w}_n) = \mathbf{0},$$

for some elements $b_1, \ldots, b_n$ of the field. Then

$$T(b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n) = \mathbf{0}, \quad \text{so} \quad b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n \in \mathrm{null}(T).$$

But then

$$b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n = a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m,$$

for some choice of scalars $a_1, \ldots, a_m$. Thus

$$a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m - b_1\mathbf{w}_1 - \cdots - b_n\mathbf{w}_n = \mathbf{0}.$$

Since the list $(\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n)$ is linearly independent,

$$a_1 = \cdots = a_m = b_1 = \cdots b_n = 0.$$

From this we conclude that $(T(\mathbf{w}_1), \ldots, T(\mathbf{w}_n))$ is linearly independent.