Name: Key

Mathematics 108A: Practice Quiz F

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Give complete answers to each of the following questions.

1. Let $\mathcal{P}_3(\mathbb{R})$ denote the space of polynomials of degree three, with basis

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2, \quad p_3(x) = x^3.$$

Suppose that $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ is the linear transformation defined by

$$T(p(x)) = \frac{d^2p}{dx}(x) + 2\frac{dp}{dx}(x).$$

What is the matrix $\mathcal{M}(T)$ of T with respect to this basis?

$$T(1) = 0 = \begin{pmatrix} 1 & x & x^2 & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x) = 2 = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x^2) = 2 + 4x = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x^3) = 6x + 6x^2 = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 6 \\ 0 \end{pmatrix}$$

$$\mathcal{M}(T, \beta, \beta) = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis for $\mathbb{R}^2$ defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 - 2x_2 \\ 4x_1 - 3x_2 \end{pmatrix},$$

what is $\mathcal{M}(T, \beta, \beta) = \mathcal{M}(T, (\mathbf{v}_1, \mathbf{v}_2), (\mathbf{v}_1, \mathbf{v}_2))$?

$$T\vec{v}_1 = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_1 = (\vec{v}_1 \ \vec{v}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T\vec{v}_2 = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -\vec{v}_2 = (\vec{v}_1 \ \vec{v}_2)\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\mathcal{M}(T, \beta, \beta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Suppose that $T \in \mathcal{L}(V, W)$. Show that T is injective if and only if $\text{null}(T) = \{\mathbf{0}\}$.

$\Rightarrow$: If $\vec{u} \in$ null(T), $T(\vec{u}) = \vec{0}$. But also $T(\vec{0}) = \vec{0}$

Since T is injective $\vec{u} = \vec{0}$. Hence null $T = \{\vec{0}\}$

$\Leftarrow$: If $\vec{u}, \vec{v} \in \vec{V}$ and $T(\vec{u}) = T(\vec{v})$, then

$T(\vec{u} - \vec{v}) = \vec{0}$ so $\vec{u} - \vec{v} \in$ null(T). Since

null$(T) = \{\vec{0}\}$ $\vec{u} - \vec{v} = \vec{0}$ & $\vec{u} = \vec{v}$

Therefore T is injective.