Name: Key

Mathematics 108A: Practice Quiz G

August 7, 2008

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1. a. Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V is linearly independent if and only if ...

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \Rightarrow a_1 = \cdots = a_n = 0$$

b. Complete the following sentence: A list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V spans a vecor space V if and only if ...

Whenever $\vec{v} \in V$, there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$

2. Prove that if V and W are two vector spaces of the same dimension, they are isomorphic

Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for V, $(\vec{w}_1, \ldots, \vec{w}_n)$ a basis for W.

Define $T: V \longrightarrow W$ by

$$T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1\vec{w}_1 + \cdots + a_n\vec{w}_n$$

T is clearly linear.

T is surjective because $(\vec{w}_1, \ldots, \vec{w}_n)$ spans W

If $T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = T(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n)$

then $a_1\vec{w}_1 + \cdots + a_n\vec{w}_n = b_1\vec{w}_1 + \cdots + b_n\vec{w}_n$

$(a_1 - b_1)\vec{w}_1 + \cdots + (a_n - w_n)\vec{w}_n = \vec{0}$

$(\vec{w}_1, \ldots, \vec{w}_n)$ linearly independent $\Rightarrow a_1 - b_1 = \cdots = a_n - b_n = 0$

$\therefore a_1 = b_1, \ldots, a_n = b_n$. This implies that T is injective

3. Prove that if $T : V \to V$ is a linear map, then T is injective if and only if T is surjective.

By Theorem 3.4,

$$n = \dim V = \dim(\text{null}(T)) + \dim(\text{range}(T)).$$

Hence $\dim(\text{null}(T)) = 0 \iff \dim(\text{range}(T)) = n$.

LEMMA. $\dim(\text{range}(T)) = n = \dim V$

$\iff \text{range}(T) = V$

$\Leftarrow$: Clearly $\text{range}(T) = V \implies \dim(\text{range}(T)) = \dim V$.

$\Rightarrow$ Note that $\text{range}(T) \subseteq V$. A basis $(\vec{v}_1, \ldots, \vec{v}_n)$ for $\text{range}(T)$ can be extended to a basis for V. Since $\dim(\text{range}(T)) = \dim V$, this extension must be trivial and $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for V.

Hence $\text{range}(T) = V$. QED

Therefore

T injective $\iff \dim(\text{null}(T)) = 0$

$\iff \dim(\text{range}(T)) = \dim V$

$\iff \text{range}(T) = V \iff T$ is surjective.