Name: Key

Mathematics 108A: Practice Quiz I

August 19, 2008

Professor J. Douglas Moore

I. Multiple Choice. Circle the best answer to each of the following questions.

1. Recall that a list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V is *linearly independent* if and only if

$$a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n = 0 \quad \Rightarrow \quad a_1 = \cdots = a_n = 0.$$

It is *linearly dependent* if it is not linearly independent. Using these definitions, one can conclude that if the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots\cdots &\cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned} \tag{1}$$

where the a_{ij}'s are known elements of the field $\mathbb{F}$, has no nontrivial solutions, then the **column** vectors of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

are

linearly independent | linearly dependent | no conclusion possible

2. If the system (1) does have nonzero solutions, one can conclude that the **column** vectors of the matrix A are

linearly independent | linearly dependent | no conclusion possible

3. Suppose that $m \neq n$. If the system (1) does have nonzero solutions, one can conclude that the **row** vectors of the matrix A are

linearly independent | linearly dependent | no conclusion possible

4. Recall that a list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V *span* V if and only if

$$\mathbf{v} \in V \quad \Rightarrow \quad \mathbf{v} = a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n,$$

for some $a_1, \ldots a_n \in \mathbb{F}$. Using this definition, one can conclude that if the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \cdots\cdots & \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{2}$$

where the a_{ij}'s are known elements of the field $\mathbb{F}$, has a solution for any choice of $b_1, \ldots, b_m \in \mathbb{F}$, then the **column** vectors of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

span $\mathbb{F}^m$ do not span $\mathbb{F}^m$ no conclusion possible

5. If the system (2) does not have a solution for some choice of $b_1, \ldots, b_m \in \mathbb{F}$, one can conclude that the **column** vectors of the matrix A

span $\mathbb{F}^m$ do not span $\mathbb{F}^m$ no conclusion possible

6. Suppose that $m \neq n$. If the system (2) has a solution for any choice of $b_1, \ldots, b_m \in \mathbb{F}$, one can conclude that the **row** vectors of the matrix A

span $\mathbb{F}^m$ do not span $\mathbb{F}^m$ no conclusion possible

7. Recall that a list of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ in V is a *basis* for V if and only if $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is linearly independent and spans V. Consider the matrix equation $A\mathbf{x} = \mathbf{0}$, where A is an $n \times n$ square matrix. If the column vectors of A are linearly independent, we can conclude that this list

is a basis for $\mathbb{F}^n$ is not a basis for $\mathbb{F}^n$ no conclusion possible

II. Complete answers. Write out complete solutions to each of the following problems.

The Main Theorem of Chapter 5 is: If V is a finite-dimensional nonzero complex vector space, every linear map $T : V \to V$ has an eigenvalue.

1. This is proven in several steps. Suppose that $\dim V = n$ and that $\mathbf{v}$ is a nonzero element of V. Show that the list of vectors

$$(\mathbf{v}, T(\mathbf{v}), \ldots, T^n(\mathbf{v}))$$

is linearly dependent

By the Comparison Theorem any linearly independent list must have length $\leq n = \dim V$. Hence $(\vec{v}, T\vec{v}, \ldots, T^n\vec{v})$ is linearly dependent.

2. Show that there is a nonzero polynomial with complex coefficients,

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad \text{such that} \quad p(T)\mathbf{v} = \mathbf{0}.$$

$\therefore$ $a_0 \vec{v} + a_1 T\vec{v} + \cdots + a_n T^n \vec{v} = 0$, for some $a_0, a_1, \ldots, a_n \in \mathbb{F}$, not all zero. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. Then $p(T)\vec{v} = \vec{0}$

3. Suppose that the degree of p is m. Use the Fundamental Theorem of Algebra to show that there exist complex constants $c, \lambda_1, \ldots, \lambda_n$ with $c \neq 0$ such that

$$p(T)\mathbf{v} = c(T - \lambda_1 I) \cdots (T - \lambda_m I)(\mathbf{v}) = \mathbf{0}.$$

If p has degree m, m is the largest integer such that $a_m \neq 0$. By the Fundamental Theorem of Algebra, $p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$, where $c \neq 0$. But then $p(T)\vec{v} = c(T - \lambda_1 I) \cdots (T - \lambda_m I)\vec{v} = \vec{0}$

4. Show that for some λ_i, the subspace $W_{\lambda_i} = \{\mathbf{v} \in V : (T - \lambda_i I)(\mathbf{v}) = \mathbf{0}\}$ is nonzero. Explain how this finishes the proof of the Main Theorem.

$(T - \lambda_1 I) \cdots (T - \lambda_m I)\vec{v} = \vec{0} \Rightarrow (T - \lambda_1 I) \cdots (T - \lambda_m I)$ is not injective
$\therefore$ $T - \lambda_i I$ is not injective for some i, $1 \leq i \leq m$.
Then $W_{\lambda_i} = \{\vec{v} \in V : (T - \lambda_i I)(\vec{v}) = \vec{0}\} \neq \{\vec{0}\}$. Hence λ_i is an eigenvalue for T.