Math 117: Infinite Sequences

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We next describe how the Bolzano-Weierstrass Theorem (which in turn is an immediate consequence of the Heine-Borel Theorem) can be used to prove the existence of limits for Cauchy sequences of real numbers. In addition to the Cauchy Sequence Theorem described below, we also present two other very useful theorems on convergence of sequences, the Monotone Convergence Theorem and the Subsequence Theorem.

As we will see at the end of §2, we can construct the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$ by taking equivalence classes of Cauchy sequences of rational numbers.

1 Convergence of infinite sequences

A *sequence* of real numbers is simply a function $s : \mathbb{N} \to \mathbb{R}$. We let s_n denote the value of the function s at $n \in \mathbb{N}$ and we sometimes use the notation (s_n) for the sequence. The key definition in the subject is:

Definition. A sequence (s_n) of real numbers is said to *converge* to a real number s if for every $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |s_n - s| < \epsilon.$$

In this case, we write $s = \lim s_n$. A sequence (s_n) of real numbers which does not converge to a real number is said to *diverge*.

Example 1. We claim that the sequence (s_n) defined by $s_n = 1/n$ converges to 0. Indeed, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $N > 1/\epsilon$ and thus $1/N < \epsilon$ by the Archimedean property of the real numbers. It follows that

$$n > N \quad \Rightarrow \quad 0 < \frac{1}{n} < \frac{1}{N} \quad \Rightarrow \quad |s_n - 0| = \left|\frac{1}{n} - 0\right| < \epsilon.$$

Using the same technique, you could show that the sequence (s_n) defined by $s_n = k/n$ converges to 0, whenever k is a real number.

Example 2. On the other hand, the sequence (s_n) defined by $s_n = 1 + (-1)^n$ diverges. We prove this by contradiction. Suppose that this sequence (s_n) were to converge to s. We could then take $\epsilon = 1$, and there would exist $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < 1.$$

But then if $n > N$ and n is even, we would have $s_n = 2$ and $s_{n+1} = 0$. Hence,

$$2 = |s_n - s_{n+1}| \leq |s_n - s| + |s - s_{n+1}| < 1 + 1 = 2,$$

a contradiction.

Example 3. Suppose we want to investigate the convergence of the sequence (s_n) defined by

$$s_n = \frac{2n+3}{n+5}.$$

We can rewrite this as

$$s_n = \frac{n+3/n}{1+5/n}.$$

By Example 1, we expect that as $n \to \infty$, $(3/n)$ and $(5/n)$ converge to zero. Thus we expect (s_n) to converge to $2/1 = 2$. To construct an argument to PROVE that this is the case we need to investigate the inequality

$$\left|\frac{2n+3}{n+5} - 2\right| < \epsilon \quad \text{or} \quad \left|\frac{2n+3-2(n+5)}{n+5}\right| < \epsilon.$$

We can rewrite this as

$$\left|\frac{-7}{n+5}\right| < \epsilon \quad \text{or} \quad \left|\frac{7}{\epsilon}\right| < n+5.$$

Using the Archimedean property of the real numbers, we choose $N \in \mathbb{N}$ so that $N > 7/\epsilon$. Then

$$n > N \quad \Rightarrow \quad n > \frac{7}{\epsilon} \quad \Rightarrow \quad \frac{7}{n} < \epsilon$$
$$\Rightarrow \quad \left|\frac{2n+3}{n+5} - 2\right| < \epsilon \quad \Rightarrow \quad |s_n - 2| < \epsilon.$$

Remark. We would also like to be able to show that the sequence (s_n) defined by

$$s_n = \frac{2n^2+3n+6}{n^2+5n+3} \tag{1}$$

converges, and with a little effort we can make a strong plausibility argument that it should converge to 2. But a proof of convergence along the lines of Example 3 would be a little lengthy. Fortunately, we can develop some convergence theorems for sequences that helps avoid the long algebraic calculations.

Proposition 1. *Suppose that (s_n) is a sequence of real numbers which converges to $s \in \mathbb{R}$. Then (s_n) is bounded.*

Proof: Choose $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < 1,$$

and let

$$M = \sup\{|s_1|, |s_2|, \ldots, |s_N|, |s|\} + 1.$$

If $n \leq N$, then $|s_n| \leq M$, while if $n \geq N+1$,

$$|s_n| \leq |s_n - s| + |s| \leq 1 + |s| \leq M.$$

Thus $|s_n| < M$ for all $n \in \mathbb{N}$, and (s_n) is bounded.

Proposition 2. *Suppose that (s_n) and (t_n) are convergent sequences of real numbers with $\lim s_n = s$ and $\lim t_n = t$. Then*

1. $(s_n + t_n)$ *converges and* $\lim(s_n + t_n) = s + t$,
2. $(s_n t_n)$ *converges and* $\lim(s_n t_n) = st$,
3. (s_n/t_n) *converges and* $\lim(s_n/t_n) = s/t$, *provided* $t_n \neq 0$ *for all* n *and* $t \neq 0$.

Before proving this let us consider the following application. Consider the sequence (s_n) defined by (1), which can be rewritten as

$$s_n = \frac{2 + 3/n + 6/n^2}{1 + 5/n + 3/n^2},$$

or equivalently,

$$s_n = \frac{t_n}{u_n} \quad \text{where} \quad t_n = 2 + \frac{3}{n} + \frac{6}{n^2} \quad \text{and} \quad u_n = 1 + \frac{5}{n} + \frac{3}{n^2}.$$

Using parts 1 and 2 of Proposition 2, it is easy to prove that (t_n) converges to 2 and (u_n) converges to 1. Thus it follows from part 3 of Proposition 2 that (s_n) converges to 2/1 or 2.

Proof of Proposition 2, Part 1: Let $\epsilon > 0$ be given. Since (s_n) converges to s, there exists an $N_1 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_1 \quad \Rightarrow \quad |s_n - s| < \frac{\epsilon}{2}.$$

Since (t_n) converges to s, there exists an $N_2 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_2 \quad \Rightarrow \quad |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then using the triangle inequality, we conclude that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is exactly what we needed to prove.

Remark. The technique used to prove Part 1 is called the $\epsilon/2$ trick. It is used repeatedly in constructing proofs of theorems about series. For example, we use it also in Part 2.

Proof of Proposition 2, Part 2: As a preliminary step, we divide $|s_n t_n - st|$ into a sum of two terms,

$$\begin{aligned} |s_n t_n - st| &= |(s_n t_n - s_n t) + (s_n t - st)| \\ &\leq |s_n t_n - s_n t| + s_n t - st| \leq |s_n||t_n - t| + |t||s_n - s|. \end{aligned} \tag{2}$$

By Proposition 1, the sequence (s_n) is bounded, which means there exists a real number M_1 such that $|s_n| \leq M_1$ for all $n \in \mathbb{N}$. If we set $M = \sup(M_1, |t|)$, then it follows from (2) that

$$|s_n t_n - st| \leq M|t_n - t| + M|s_n - s|. \tag{3}$$

Now we proceed just as in Part 1. Let $\epsilon > 0$ be given. There exists an $N_1 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_1 \quad \Rightarrow \quad |s_n - s| < \frac{\epsilon}{2M},$$

and there exists an $N_2 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_2 \quad \Rightarrow \quad |t_n - t| < \frac{\epsilon}{2M}.$$

Let $N = \max(N_1, N_2)$. Then it follows from (3) that

$$|s_n t_n - st| \leq M|t_n - t| + M|s_n - s| \leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon,$$

which is exactly what we needed to show.

Proof of Proposition 2, Part 3: See page 167 of the text [1].

Fundamental Problem of Analysis. Various algorithms will provide a sequence (s_n) of real numbers. The problem that often arises is that of showing that the sequence converges to some limit. For example, suppose we consider the sequence (s_n) defined by

$$s_n = \left(1 + \frac{1}{n}\right)^n. \tag{4}$$

We would like to show that this sequence converges. Then we can call the limit of the sequence e.

The simplest case in which this problem can be solved is that of monotone sequences. A sequence (s_n) of real numbers is said to be *increasing* if $s_n \leq s_{n+1}$ for all n. It is *decreasing* if $s_n \geq s_{n+1}$ for all n. A sequence (s_n) of real numbers is *monotone* if it is either increasing or decreasing.

Monotone Convergence Theorem. *A bounded monotone sequence* (s_n) *of real numbers must converge.*

Proof: We first consider the case in which (s_n) is increasing. By hypothesis, $S = \{s_n : n \in \mathbb{N}\}$ is bounded. By the completeness axiom, it must therefore have a supremum s. Let $\epsilon > 0$. Since $s = \sup S$, there exists $N \in \mathbb{N}$ such that $s_N > s - \epsilon$. Then

$$n > N \quad \Rightarrow \quad s_n \geq s_N > s - \epsilon \quad \text{and} \quad s_n \leq s \quad \Rightarrow \quad |s_n - s| < \epsilon.$$

Hence $s = \lim s_n$.

The case where (s_n) is decreasing is proven in a similar fashion.

Definition. A sequence (s_n) of real numbers is said to *diverge* to ∞ (and we write $\lim s_n = \infty$) if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad s_n > M.$$

A sequence (s_n) of real numbers is said to *diverge* to $-\infty$ (and we write $\lim s_n = -\infty$) if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad s_n < M.$$

It is not difficult to show that if an increasing sequence (s_n) is not bounded, the sequence diverges to ∞. Similarly, if a decreasing sequence (s_n) is not bounded, the sequence diverges to $-\infty$.

2 Cauchy sequences

It is often useful to consider sequences from more general spaces than just $\mathbb{R}$. A sequence of elements from a metric space (X, d) is just a function $x : \mathbb{N} \to X$; we will denote such a sequence by (x_n).

Definition. A sequence (x_n) of elements in a metric space (X, d) is said to *converge* to an element $x \in X$ if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad d(x_n, x) < \epsilon.$$

A sequence (x_n) of elements in (X, d) is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad d(x_n, x_m) < \epsilon.$$

For us, the most important cases are the cases in which $X = \mathbb{R}$ or $X = \mathbb{R}^n$. In the case of $\mathbb{R}^n$,

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|.$$

Note that a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^n$ is a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < \epsilon.$$

Cauchy Theorem. *A Cauchy sequence in* $\mathbb{R}^n$ *must converge.*

Note. It is not true that a Cauchy sequence in an arbitrary metric space is convergent. It is true in $\mathbb{R}$ or in $\mathbb{R}^n$ with the standard metric, because the Heine-Borel theorem and the Bolzano-Weierstrass Theorem are available.

Proof: We first prove a lemma:

Lemma. *If* $(\mathbf{x}_n)$ *is a Cauchy sequence in* $\mathbb{R}^n$*, then*

$$S = \{\mathbf{x}_n : n \in \mathbb{N}\}$$

is bounded.

Proof of Lemma: Use the fact that $(\mathbf{x}_n)$ is a Cauchy sequence to choose $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < 1.$$

We then let

$$M = \sup\{|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_N|, |\mathbf{x}_{N+1}|\} + 1.$$

If $n \leq N$, then $|\mathbf{x}_n| \leq M$, while if $n \geq N + 1|$,

$$|\mathbf{x}_n| \leq |\mathbf{x}_n - \mathbf{x}_{N+1}| + |\mathbf{x}_{N+1}| \leq 1 + |\mathbf{x}_{N+1}| \leq M.$$

Thus $|\mathbf{x}_n| < M$ for all $n \in \mathbb{N}$.

Proof of Cauchy Sequence Theorem: Suppose that $(\mathbf{x}_n)$ and let

$$S = \{\mathbf{x}_n : n \in \mathbb{N}\}.$$

We divide into two cases.

Case I: S is finite. In this case, we can set

$$\epsilon = \min\{|\mathbf{x}_n - \mathbf{x}_m)| : \mathbf{x}_n \neq \mathbf{x}_m,\ n, m \in \mathbb{N}\},$$

a positive number, since there are only finitely many distances between distinct elements of S. By definition of Cauchy sequence, there exists an element $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < \epsilon.$$

But this implies that $\mathbf{x}_n = \mathbf{x}_m$ for $n, m > N$, and hence the Cauchy sequence converges.

Case II: S is infinite. In this case, S is infinite and bounded by the Lemma, so it follows from the Bolzano-Weierstrass Theorem that S has an accumulation point $\mathbf{x}$. We claim that the Cauchy sequence $(\mathbf{x}_n)$ converges to $\mathbf{x}$.

To prove the claim, we note that there exists $N \in \mathbb{N}$ such that

$$n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < \epsilon/2.$$

Since $\mathbf{x}$ is an accumulation point, infinitely many points of S must lie in the deleted neighborhood $N^*(\mathbf{x}; \epsilon/2)$. Thus there exists an $n > N$ such that $\mathbf{x}_n \in N(\mathbf{x}; \epsilon/2)$. Then

$$m \in \mathbb{N} \quad \text{and} \quad m > N \quad \Rightarrow \quad |\mathbf{x}_m - \mathbf{x}| < |\mathbf{x}_m - \mathbf{x}_n| + |\mathbf{x}_n - \mathbf{x}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $(\mathbf{x}_n)$ converges to $\mathbf{x}$, verifying the claim and finishing the proof of the Theorem.

Many algorithms in analysis produce Cauchy sequences, so the preceding theorem is extremely useful.

We say that a metric space (X, d) is *complete* if every Cauchy sequence in X converges. Thus the above theorem states that $(\mathbb{R}^n, d)$ is complete when d is the standard metric on $\mathbb{R}^n$.

Digression: Constructing the reals from the rationals. Suppose we have constructed the rationals $\mathbb{Q}$ via set theory but want to develop real numbers from scratch without assuming that a complete ordered field exists. Cauchy sequences provide the technique needed for doing this. We say that a sequence (x_n) of rational numbers is a *Cauchy sequence of rational numbers* if for every rational number ϵ such that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |x_n - x_m| < \epsilon.$$

Two such sequences (x_n) and (y_n) are said to be *equivalent* for every rational number ϵ such that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |x_n - y_n| < \epsilon.$$

If two Cauchy sequence of rational numbers (x_n) and (y_n) are equivalent we write $(x_n) \sim (y_n)$. We can show that $\sim$ is an equivalence relation. The set of equivalence classes is then identified with the real numbers $\mathbb{R}$.

Thus for example, the decimal expansion of π provides a Cauchy sequence of rational numbers

$$x_1 = 3, \quad x_2 = 3.1, \quad x_3 = 3.14, \quad x_4 = 3.141, \quad x_5 = 3.1415, \quad x_6 = 3.14159, \quad x_7 = 3.141592, \quad \ldots$$

and its equivalence class will be the real number π.

We identify an element $x \in \mathbb{Q}$ with the Cauchy sequence (x_n) of rational numbers such that $x_n = x$ for all $n \in \mathbb{N}$, thereby realizing $\mathbb{Q}$ as a subset of $\mathbb{R}$. We can next define addition, multiplication and order of equivalence classes, and check that these operations satisfy the axioms for complete ordered fields. When we do all this, we find that if (x_n) is a Cauchy sequence of rational numbers, its limit is just its equivalence class. Of course, the details of the construction are lengthy and there are many details to check. But these techniques are very useful to analysts when studying completions of metric spaces.

3 Subsequences

Suppose that (s_n) is a sequence of real numbers. If $n_1 < n_2 < \cdots < n_k < \dots$ is an increasing sequence of natural numbers, then the sequence (t_k) where $t_k = s_{n_k}$ is called a *subsequence* of (s_n). For example, suppose that $s_n = (-1)^n$, a sequence that we know diverges. Then we can define a subsequence (t_k) by $t_k = s_{2k}$. Then $t_k = 1$ for all k so this subsequence (t_k) does converge.

Even though a bounded sequence of real numbers need not converge, we do have the following remarkable fact regarding subsequences:

Subsequence Theorem. *A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

As in the Cauchy theorem, the proof rests on the Bolzano-Weierstrass Theorem. Suppose that $(\mathbf{x}_n)$ is a bounded sequence in $\mathbb{R}^n$. We divide into two cases.

Case I: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ **is finite.** In this case, $\mathbf{x}_n$ must be a fixed element $\mathbf{x}_0$ of S for infinitely many $n \in \mathbb{N}$. We can let n_1 be the smallest element of $\mathbb{N}$ such that $\mathbf{x}_{n_1} = \mathbf{x}_0$, n_2 be the second smallest element of $\mathbb{N}$ such that $\mathbf{x}_{n_2} = \mathbf{x}_0$, and so forth. We thereby obtain a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ which converges to $\mathbf{x}_0$.

Case II: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ **is infinite.** Since S is bounded, it follows from the Bolzano-Weierstrass Theorem that S has an accumulation point $\mathbf{x}$.

We choose n_1 so that $\mathbf{x}_{n_1} \in N^*(\mathbf{x}; 1)$ and let

$$\epsilon_1 = \min\left(|\mathbf{x}_{n_1} - \mathbf{x}|, \frac{1}{2}\right).$$

We next choose n_2 so that $\mathbf{x}_{n_2} \in N^*(\mathbf{x}; \epsilon_1)$ and let

$$\epsilon_2 = \min\left(|\mathbf{x}_{n_2} - \mathbf{x}|, \left(\frac{1}{2}\right)^2\right).$$

Continuing in this fashion, we obtain a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ such that

$$|\mathbf{x}_{n_k} - \mathbf{x}| < \left(\frac{1}{2}\right)^k.$$

The subsequence $(\mathbf{x}_{n_k})$ converges to $\mathbf{x}$.

Definition. Let (s_n) be a sequence of real numbers, and let

$$M_n = \sup\{s_n, s_{n+1}, s_{n+2}, \dots\}, \quad m_n = \inf\{s_n, s_{n+1}, s_{n+2}, \dots\}.$$

Then (M_n) is a monotone decreasing sequence, while (m_n) is a monotone increasing sequence. We let

$$\limsup(s_n) = \lim M_n, \qquad \liminf(s_n) = \lim m_n.$$

Note that $\liminf(s_n) \leq \limsup(s_n)$, with equality holding if and only if (s_n) converges.

Proposition. Suppose that (s_n) is a sequence of real numbers and let

$$S = \{ \text{ all limits of subsequences of } (s_n) \ \}.$$

Then $\sup S = \limsup(s_n)$ and $\inf S = \liminf(s_n)$.

The proof is beyond the scope of the course; we refer to [1], §19 for many examples.

4 Infinite series

Definition. An *infinite series* is a sum of the form

$$\sum_{n=0}^{\infty} a_n,$$

where the a_n's are real numbers. The infinite series is said to *converge* to a real number s if the partial sum

$$s_n = \sum_{m=0}^{n} a_m$$

converges to s. In this case, we write

$$s = \sum_{n=0}^{\infty} a_n.$$

An infinite series which does not converge to a real number is said to *diverge*.

Example. One of the most important infinite series is the *geometric series*

$$\sum_{n=0}^{\infty} x^n, \quad \text{where} \quad |x| < 1.$$

In this case, we have the partial sum

$$s_n = \sum_{m=0}^{n} x^m = 1 + x + \cdots + x^n.$$

Since

$$xs_n = x + x^2 + \cdots + x^{n+1},$$

we find that

$$s_n - xs_n = 1 - x^{n+1}, \quad \text{or} \quad s_n = \frac{1 - x^{n+1}}{1 - x}.$$

Under the assumption that $0 < x < 1$, $\lim x^{n+1} = 0$, from which one can conclude that

$$\lim s_n = \frac{1}{1-x} \quad \text{or} \quad \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}.$$

Defining exponentials. One way of defining the real number e is by means of the infinite series,

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

To see that this infinite series converges, one can compare its partial sums with those of a geometric series,

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \leq 3.$$

We see that the increasing sequence (s_n) is bounded by the sum of the geometric series. It therefore follows from the Monotone Convergence Theorem that (s_n) is a convergent series. We are therefore justified in letting e denote the limit,

$$e = \sum_{m=0}^{\infty} \frac{1}{m!}. \tag{5}$$

Theorem. $\lim (1 + (1/n))^n = e$.

In other words, we could define e by either (5) or as the limit of the sequence (s_n) defined by (4).

Our proof follows Rudin [2], 3.31. We let

$$s_n = \sum_{m=0}^{n} \frac{1}{m!} \quad \text{and} \quad t_n = \left(1 + \frac{1}{n}\right)^n.$$

Then by the binomial theorem

$$\begin{aligned} t_n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ &\qquad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

It follows $t_n \leq t_{n+1}$, so that (t_n) is an increasing sequence, and it also follows from the above expressions that $t_n \leq s_n$. Hence (t_n) is a bounded increasing sequence and $\lim t_n$ exists and $\lim t_n \leq e$.

On the other hand, if $n \geq m$,

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

If we fix m and let $n \to \infty$, we obtain

$$\lim t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}.$$

Letting $m \to \infty$ now yields $\lim t_n \geq e$. Thus t_n converges and its limit is e.

More generally, if $x \in \mathbb{R}$, we can define e^x by means of the infinite power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{6}$$

To see that this infinite series converges, we find it useful to make use of the following:

Proposition. *Suppose that* $|a_n| \leq b_n$*, where* $b_n > 0$*, for each* $n \in \mathbb{N}$*. If* $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*

Proof: To prove this, suppose that $\sum_{n=0}^{\infty} b_n$ converges, and let $\epsilon > 0$ be given. Then

$$\exists N \in \mathbb{N} \quad \text{such that} \quad n > m > N \quad \Rightarrow \quad \sum_{k=m}^{n} b_k < \epsilon.$$

It follows that when $n > m > N$,

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k < \epsilon.$$

Thus if

$$s_n = \sum_{k=0}^{n} a_k, \quad n > m > N \quad \Rightarrow \quad |s_n - s_m| < \epsilon,$$

which implies that (s_n) is a Cauchy sequence. By the Cauchy Theorem, (s_n) converges and hence $\sum_{n=0}^{\infty} a_n$ converges.

We can apply this to show that this series (6) converges for $|x| < M$, where $M \in \mathbb{N}$ is given. Indeed, we can write

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{M-1} \frac{x^n}{n!} + x^M \sum_{k=0}^{\infty} a_k, \quad \text{where} \quad a_k = \frac{x^k}{(M+k)!}.$$

We can compare the last sum on the right with

$$\sum_{k=0}^{\infty} c_k, \quad \text{where} \quad c_k = \frac{1}{(M)!}\left(\frac{|x|}{M}\right)^k,$$

a constant multiple of a convergent geometric series with positive terms. The preceding proposition shows that the series (6) converges.

By modifying the proof of the preceding theorem, one could now establish the following limit:

$$\lim \left(1 + \frac{x}{n}\right)^n = e^x, \quad \text{for} \quad x \geq 0.$$

Moreover, by expanding the power series on the two sides, one could prove the important formula

$$e^{x+y} = e^x e^y.$$

5 The Contraction Mapping Theorem*

We now discuss a slightly more advanced topic, that some readers may want to skip on a first reading. Recall that a metric space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) converges. Thus, for example, it follows from the Cauchy Sequence Theorem that $\mathbb{R}$ and $\mathbb{R}^n$ are complete when they are given the usual metric.

A function $f : X \to X$ is called a *contraction* if for all $x, y \in X$,

$$d(f(x), f(y)) < \alpha d(x, y), \qquad \text{where} \qquad 0 < \alpha < 1. \tag{7}$$

A point $x \in X$ is said to be a *fixed point* of the contraction if $f(x) = x$. Often, one uses contractions to construct Cauchy sequences. In fact, here is one of the most useful applications of Cauchy sequences:

Contraction Mapping Theorem. *If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point.*

Sketch of proof: We start by picking a point $x_0 \in X$, and for $n \in N$, let $x_n = f(x_{n-1})$. This gives a sequence of points (x_n) in X. Suppose that $d(x_0, x_1) = \beta$. Then it follows from (7) that

$$d(x_1, x_2) < \alpha\beta, \quad d(x_2, x_3) < \alpha^2\beta, \quad \ldots, \quad d(x_n, x_{n+1}) < \alpha^n\beta, \quad \ldots.$$

Hence if $k > 0$,

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots d(x_{n+k-1}, x_{n+k}) \\ &< \alpha^n\beta(1 + \alpha + \cdots + \alpha^{k-1}) < \alpha^n \frac{\beta}{1-\alpha}, \end{aligned}$$

where we have used the formula for the sum of a geometric series. Given $\epsilon > 0$, we can choose N sufficiently large that when $n > N$,

$$\alpha^n < \epsilon \frac{1-\alpha}{\beta} \quad \text{and hence} \quad \alpha^n \frac{\beta}{1-\alpha} < \epsilon.$$

Hence for $n, m > N$, $d(x_n, x_m) < \epsilon$ and (x_n) is a Cauchy sequence.

Since (X, d) is complete, the Cauchy sequence (x_n) converges to an element $x \in X$.

Let $\epsilon > 0$ be given. If $N \in \mathbb{N}$ is sufficiently large,

$$n > N \quad \Rightarrow \quad d(x_n, x) < \frac{\epsilon}{2} \quad \Rightarrow \quad d(f(x_n), f(x)) < \alpha\frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Hence

$$m > N+1 \quad \Rightarrow \quad d(x_m, f(x)) < \frac{\epsilon}{2}, \quad \text{so} \quad d(x, f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we see that $d(x, f(x)) = 0$ and x must be a fixed point of f.

Finally, if x and y are two fixed points of f, then it follows from (7) that

$$d(x, y) = d(f(x), f(y)) < \alpha d(x, y) \quad \Rightarrow \quad x = y,$$

so the fixed point is unique.

References

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[2] W. Rudin, *Principles of mathematical analysis*, Third edition, McGraw-Hill, New York, 1976.