Math 5AI: Project 7
Mechanical systems with many degrees of freedom
(Revised version)

February 17-19, 2010

A. Mechanical systems

The Theorem of Project 6 can sometimes be used to help analyze systems of differential equations of the form

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}, \tag{1}$$

where A is an $n \times n$ symmetric matrix, $A^T = A$. Such systems of differential arise for example, in treating mechanical systems of weights connected to springs in a complicated array.

For example, consider a mechanical system in which we have two carts moving along a friction-free track, each containing containing a unit of mass and attached together by three springs, the two outer springs having spring constant k_1, the inner spring having spring constant k_2. Let

$$\begin{aligned}
x_1(t) &= \text{the position of the first cart to the right of equilibrium,} \\
x_2(t) &= \text{the position of the second cart to the right of equilibrium,} \\
F_1 &= \text{force acting on the first cart,} \\
F_2 &= \text{force acting on the second cart,}
\end{aligned}$$

with positive values for F_1 or F_2 indicating that the forces are pulling to the right.

In this simple case, it is possible to reason directly via Hooke's law that the forces F_1 and F_2 must be given by the formulae

$$F_1 = -k_1x_1 + k_2(x_2 - x_1), \qquad F_2 = -k_1x_2 + k_2(x_1 - x_2),$$

but it becomes difficult to determine the forces for more complicated mechanical systems consisting of many weights and springs. Principles from physics can simplify the procedure for finding the forces acting in such mechanical systems.

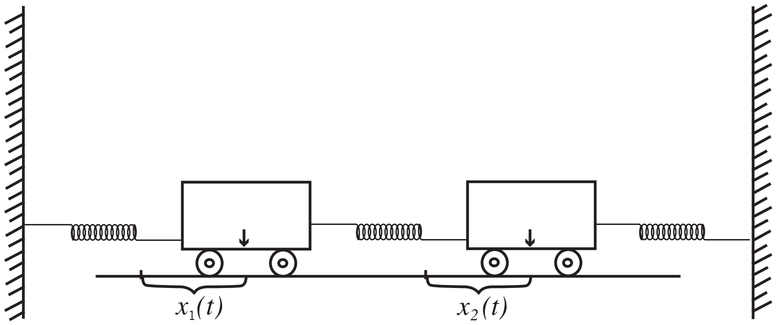

Figure 1: Two carts attached to the walls by means of springs.

The easiest calculation of the forces is based upon the notion of *work*. On the one hand, the work required to pull a weight to a new position is equal to the increase in potential energy imparted to the weight. On the other hand, we have the equation

$$\text{Work} = \text{Force} \times \text{Displacement}.$$

It follows from these two facts that if $V(x_1, x_2)$ is the potential energy of the configuration when the first cart is located at the point x_1 and the second cart is located at the point x_2, then the forces are given by the formulae

$$F_1 = -\frac{\partial V}{\partial x_1}, \qquad F_2 = -\frac{\partial V}{\partial x_2};$$

here

$$\frac{\partial V}{\partial x_1}$$

represents the derivative of V with respect to x_1 when x_2 is held fixed; thus, for example,

$$\frac{\partial}{\partial x_1}\left((\sin x_1)x_2^2\right) = (\cos x_1)x_2^2.$$

In our case, the potential energy V is the sum of the potential energies stored in each of the three springs,

$$V(x_1, x_2) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_1 x_2^2,$$

and hence we can obtain the formulae claimed before:

$$\begin{array}{rcccl} F_1 & = & -\frac{\partial V}{\partial x_1} & = & -k_1 x_1 + k_2(x_2 - x_1), \\ F_2 & = & -\frac{\partial V}{\partial x_2} & = & -k_1 x_2 + k_2(x_1 - x_2). \end{array}$$

It now follows from Newton's second law of motion that

$$\frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2,$$

$$\frac{d^2x_2}{dt^2} = -k_1x_2 + k_2(x_1 - x_2) = k_2x_1 - (k_1 + k_2)x_2.$$

We can write this system in matrix form as

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_1 + k_2) \end{pmatrix}. \tag{2}$$

1. a. To solve the above system, we need to find the eigenvalues of the matrix (2). Since this is a symmetric matrix, we know that its eigenvalues must be real. Show that the eigenvalues are

$$\lambda_1 = -k_1, \quad \text{and} \quad \lambda_2 = -(k_1 + 2k_2).$$

b. Find a basis for each of the eigenspaces V_{-k_1} and $V_{-k_1-2k_2}$.

c. Find unit length eigenvectors which lie in each of the eigenspaces.

d. Find an orthogonal matrix B of determinant one such that $B^TAB = B^{-1}AB$ is diagonal.

e. If we define new coordinates (y_1, y_2) by setting

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

our system of differential equations transforms to

$$\begin{aligned} d^2y_1/dt^2 &= -k_1y_1, \\ d^2y_2/dt^2 &= -(k_1 + 2k_2)y_2. \end{aligned}$$

We set $\omega_1 = \sqrt{k_1}$ and $\omega_2 = \sqrt{k_1 + 2k_2}$, so that this system assumes the familiar form

$$\begin{aligned} d^2y_1/dt^2 + \omega_1^2y_1 &= 0, \\ d^2y_2/dt^2 + \omega_2^2y_2 &= 0, \end{aligned}$$

a system of two noninteracting harmonic oscillators. What is the general solution to the transformed system?

f. Write down the general solution to the system of differential equations in the original coordinates (x_1, x_2). Note that it is a superposition of two *modes* of oscillation, the frequencies of oscillation being $\omega_1/(2\pi)$ and $\omega_2/(2\pi)$.

B. Many degrees of freedom

Warning! We are now entering somewhat deeper waters. Try to understand the overall picture, even if it might be difficult to remember how to reproduce the details.

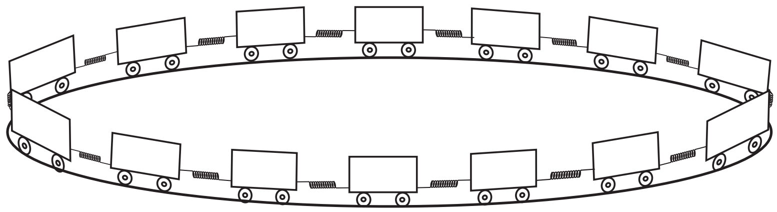

Figure 2: A circular train of carts.

Using the same approach as before, we can derive the motion of more complicated systems consisting of many masses and springs. For example, we could consider the box spring underlying the mattress in a bed. Although such a box spring contains hundreds of individual springs, and hence the matrix A in the corresponding dynamical system constains hundreds of rows and columns, it is still possible to use symmetries in the box spring to simplify the calculations, and make the problem of determing the "natural frequencies of vibration" of the mattress into a manageable problem.

We illustrate with a somewhat simpler problem, a system of n carts containing identical weights of mass m and connected by identical springs of spring constant k, moving along a circular friction-free track.

Derivation of equations for the circular train: Let x_i denote the position of the i-th cart to the right of its equilibrium position for $1 \leq i \leq n$. The potential energy stored in the springs is

$$V(x_1, \ldots, x_n) = \frac{1}{2}k(x_n - x_1)^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \ldots + \frac{1}{2}k(x_n - k_{n-1})^2$$

$$= k\Sigma_{i=1}^n x_i^2 - kx_1x_n - k\Sigma_{i=1}^{n-1} x_i x_{i+1}$$

$$= -\frac{1}{2}k \begin{pmatrix} x_1 & x_2 & x_3 & \cdot & x_n \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & \cdot & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{pmatrix}.$$

We can write this as

$$V(x_1, \ldots, x_n) = -\frac{1}{2}k\mathbf{x}^T A\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 & 0 & \cdot & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix}.$$

The force acting on the i-th cart can be calculated as minus the derivative of the potential energy with respect to the position of the i-th cart, the other carts being held fixed:

$$F_i = -\frac{\partial V}{\partial x_i}.$$

If we denote the components of the matrix A by a_{ij}, so $A = (a_{ij})$, we can express the potential energy in summation notation as

$$V(x_1, \ldots, x_n) = -\frac{1}{2}k \sum_{i,j=1}^{n} a_{ij}x_i x_j,$$

and we can carry out the calculation of the partial derivatives, obtaining

$$F_i = -\frac{\partial V}{\partial x_i} = -k \sum_{i,j=1}^{n} a_i j x_j,$$

which is equivalent to the expression one could obtain directly from Hooke's law:

$$F_i = \begin{cases} k[(x_2 - x_1) - (x_1 - x_n)], & \text{if } i = 1, \\ k[(x_{i+1} - x_i) - (x_i - x_{i-1})], & \text{if } 2 \leq i \leq n-1, \\ k[(x_2 - x_1) - (x_1 - x_n)], & \text{if } i = 1. \end{cases}$$

In vector and matrix notation, the equation for force reads

$$\mathbf{F} = kA\mathbf{x}. \tag{3}$$

On the other hand, by Newton's second law of motion,

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F},$$

and substitution into (3) yields

$$m\frac{d^2\mathbf{x}}{dt^2} = kA\mathbf{x} \quad \text{or} \quad \frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}A\mathbf{x},$$

where

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdot & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix}. \tag{4}$$

The dynamical system for the circular train of identical carts: We would like to be able to find the general solution to the vector differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}A\mathbf{x}, \tag{5}$$

where A is the symmetric matrix defined by (4). To solve this system, we need to find the eigenvalues and eigenvectors of A. Although this would be difficult for an arbitrary $n \times n$ matrix, the symmetry of the situation makes the problem quite manageable.

To simplify the calculation of the eigenvalues of the matrix A, we make use of the fact that the carts have identical properties—if we relabel the carts, shifting them to the right by one, the dynamical system remains unchanged. Thus we can define new coordinates $(z_1, \ldots, z_n)$ by setting

$$x_1 = z_2, \quad x_2 = z_3, \quad \ldots, \quad x_{n-1} = z_n, \quad x_n = z_1,$$

or in matrix terms,

$$\mathbf{x} = R\mathbf{z}, \quad \text{where} \quad R = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We can think of the transformation $\mathbf{z} \mapsto R\mathbf{z}$ as a rotation through one unit around the circular track. The new coordinates $(z_1, \ldots, z_n)$ must satisfy exactly the same system of equations, namely

$$\frac{d\mathbf{z}}{dt} = \frac{k}{m} A\mathbf{z}.$$

2. a. Show that $AR = RA$, or in other words, that he matrices A and R *commute*. Hint: you can do this by a direct calculation if you want, but there is also a more conceptual way based upon the fact that $\mathbf{x}$ and $\mathbf{y}$ satisfy the same system of differential equations

$$\frac{d\mathbf{x}}{dt} = \frac{k}{m} A\mathbf{x} \quad \text{and} \quad \frac{d\mathbf{z}}{dt} = \frac{k}{m} A\mathbf{z}.$$

This implies that

$$R^{-1}AR\mathbf{z} = A\mathbf{z}.$$

Since A and B are constant matrices and $\mathbf{z}$ can be any choice of initial conditions for the mechanical system,

$$R^{-1}AR = A \quad \text{or} \quad AR = RA.$$

b. Our next goal is to find the eigenvalues of R. Show that if $\mathbf{x}$ is an eigenvector for R corresponding to the eigenvalue λ, the components of $\mathbf{x}$ must satisfy the equations

$$x_2 = \lambda x_1, \quad x_3 = \lambda x_2, \quad \ldots, x_n = \lambda x_{n-1}, \quad x_1 = \lambda x_n, \tag{6}$$

and hence $x_i = \lambda^n x_i$ for each i, $1 \leq i \leq n$. If $\mathbf{x}$ is a nonzero eigenvector for R, at least one $x_i \neq 0$, and hence

$$\lambda^n = 1. \tag{7}$$

Fortunately, this equation can be solved via Euler's formula. Verify that the eigenvalues are

$$\lambda = \eta^j, \quad \text{for } 1 \leq j \leq n,$$

$$\text{where} \quad \eta = e^{2\pi i/n} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right). \tag{8}$$

(Terminology: We say that η is a primitive n-th root of unity.) Use the expression for the eigenvalues to show that for $1 \leq j \leq n$,

$$\mathbf{e}_j = \begin{pmatrix} 1 \\ \eta^j \\ \eta^{2j} \\ \cdot \\ \eta^{(n-1)j} \end{pmatrix},$$

is an eigenvector for R. Since the n eigenvalues are all distinct, each eigenspace is one-dimensional.

c. Use the fact that $AR = RA$ to show that the eigenvectors for R are also eigenvectors for A. Moreover, since the eigenspaces of R are all one-dimensional, $A\mathbf{e}_j$ must be a multiple of $\mathbf{e}_j$; in other words,

$$A\mathbf{e}_j = \lambda_j \mathbf{e}_j, \quad \text{for some number } \lambda_j.$$

Show that

$$\lambda_j = -2 + 2\cos(2j\pi/n).$$

Thus it follows from the familiar trigonometry formula

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

that

$$\lambda_j = -2 + 2[\cos(j\pi/n)]^2 - 2[\sin(j\pi/n)]^2 = -4[\sin(j\pi/n)]^2.$$

Note that $\lambda_j = \lambda_{n-j}$, and hence the eigenspaces for A are two-dimensional, except for the eigenspace corresponding to $\lambda_n = 0$ and $\lambda_{n/2} = 1$ when n is even.

Conclusion: By the theorem from the previous project, there is an $n \times n$ orthogonal matrix B of determinant one such that

$$B^{-1}AB = \begin{pmatrix} -4[\sin(\pi/n)]^2 & 0 & \cdots & 0 \\ 0 & -4[\sin(2\pi/n)]^2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus if we let

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{pmatrix},$$

the system of equations becomes

$$\frac{d^2y_1}{dt^2} = -4\frac{k}{m}[\sin(\pi/n)]^2 y_1, \ \frac{d^2y_2}{dt^2} = -4\frac{k}{m}[\sin(\pi/n)]^2 y_2, \cdots, \frac{d^2y_n}{dt^2} = 0.$$

Thus the frequencies of vibration of the mechanical system are

$$\frac{\omega_j}{2\pi}, \quad \text{where} \quad \omega_j = 2\sqrt{\frac{k}{m}}\left|\sin\left(\frac{j\pi}{n}\right)\right|.$$

Note that $\omega_n = 0$ corresponding to a constant speed motion of the carts around the track.

3. Notice that we used complex numbers to find the solution to (5), even though the eigenvalues and eigenspaces should be real. Show that

$$\mathbf{e}_j + \mathbf{e}_{n-j} \quad \text{and} \quad i(\mathbf{e}_j - \mathbf{e}_{n-j})$$

form a *real* basis for the eigenspace corresponding to the eigenvalue $\lambda_j = \lambda_{n-j}$. When n is even, all of the eigenspaces have dimension two.

Homework 7. Due Friday, February 19, 2010.

H.7.1.a. Find the eigenvalues of the 3×3 matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix},$$

and a basis for each of the eigenspaces for A.

b. Find a 3×3 matrix B such that $B^T B = I$ and $\det B = 1$ such that

$$B^T AB = B^{-1}AB$$

is diagonal.

c. Use the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

to put the system of equations

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x} \tag{9}$$

into diagonal form, and solve the resulting system.

d. Find the general solution to (9) in the original coordinates (x_1, x_2, x_3). What are the frequencies of oscillation?