Math 5AI: Project 9
Linear transformations

March 1, 2010

Abstract Definition. A *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function (or mapping) $T : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies

$$T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}), \quad \text{for all } a, b \in \mathbb{R} \text{ and all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Actually, it turns out that any linear transformation is just multiplication by a matrix with real entries. Suppose that we have an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{1}$$

Concrete Definition. The *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by the matrix A is the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T_A(\mathbf{x}) = A\mathbf{x}, \quad \text{for } \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

Ultimately, the abstract definition extends to allow us to define linear transformations between other vector spaces, not just $\mathbb{R}^n$ and $\mathbb{R}^m$, an idea that we will return to later.

Sometimes linear transformations represent geometric motions. For example, the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

represents a counterclockwise rotation in the plane through an angle θ.

Sometimes linear transformations are used to represent homogeneous linear systems of equations. For example, if A is the matrix given by (1), the solutions to the linear system

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0, \\ \cdots\cdots & . & . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0, \end{array} \tag{3}$$

are simply the vectors $\mathbf{x} \in \mathbb{R}^n$ which go to zero under the linear transformation T_A defined by (2).

Definition. The *null space* or the *kernel* of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the linear subspace of $\mathbb{R}^n$ defined by

$$\mathrm{null}(T) = \ker(T) = \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\}.$$

1. a. Find a basis for the null space of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}.$$

(Hint: Use the elementary row operations to reduce A to row-reduced echelon form. This changes the linear transformation, but **does not change the null space.**)

b. Find a basis for the null space of the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & -5 \end{pmatrix}.$$

c. Find a basis for the null space of the linear map $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & -2 & 0 & -2 & -1 \\ 0 & 0 & 1 & -5 & -1 \\ 1 & -2 & 1 & -7 & -2 \end{pmatrix}.$$

d. Find a basis for the null space of the linear map $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 3 & 1 & 2 & -3 \\ 1 & 3 & 2 & 5 & -4 \\ 2 & 6 & 3 & 7 & -7 \end{pmatrix}.$$

Note that if we set

$$\begin{array}{rcl} \mathbf{a}_1 & = & (a_{11}, a_{12}, \ldots, a_{1n}), \\ \mathbf{a}_2 & = & (a_{21}, a_{12}, \ldots, a_{2n}), \\ \cdot & \cdot & \cdots \\ \mathbf{a}_m & = & (a_{m1}, a_{m2}, \ldots, a_{mn}), \end{array}$$

we can write the system (3) as

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \mathbf{a}_2 \cdot \mathbf{x} = 0, \cdots, \mathbf{a}_m \cdot \mathbf{x} = 0.$$

In other words, $\mathrm{null}(T)$ is just the set of all vectors which are perpendicular to the rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of the matrix A. Indeed, if we let

$$W = \mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m),$$

then null(T) is just the linear subspace of all vectors which are orthogonal to all vectors of W. We can say that null(T) is the *orthogonal complement* to W in $\mathbb{R}^5$.

2. a. Find a basis for the null space of the linear map $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 3 & 1 & 2 & -3 \\ 1 & 3 & 2 & 5 & -4 \\ 2 & 6 & 3 & 7 & -6 \end{pmatrix}.$$

b. Find a basis for the subspace of $\mathbb{R}^5$ which is spanned by the rows of A.

(Hint: Use the SAME elementary row operations in each case, but use the resulting row-reduced echelon form differently.)

Now we go back and consider the nonhomogeneous linear system

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2, \\ \cdots\cdots & . & . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m. \end{array} \tag{4}$$

Using the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by (2), we can write this linear system as

$$T_A(\mathbf{x}) = \mathbf{b}.$$

Definition. The *range* of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is

$$\text{range}(T) = \{\mathbf{b} \in \mathbb{R}^m : T(\mathbf{x}) = \mathbf{b} \text{ for some } \mathbf{x} \in \mathbb{R}^n.\}.$$

One can check that range(T) is a linear subspace of $\mathbb{R}^m$.

3. Is the basis for the range of the linear transformation T_A the same thing as a basis for the columns of the matrix A?

4. a. Find a basis for the range of the linear map $T : \mathbb{R}^1 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Hint: This should be virtually immediate.

b. Find a basis for the range of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ -2 & 0 \\ 0 & 1 \\ -2 & -5 \end{pmatrix}.$$

Hint: Use the elementary **column** operations to reduce A to column-reduced echelon form (if it isn't already in that form).

c. Find a basis for the range of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & -2 & -3 \end{pmatrix}.$$

d. Find a basis for the range of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^5$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 5 \\ 0 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & 2 & 3 \end{pmatrix}.$$

Definition. If V is a linear subspace of $\mathbb{R}^n$, the *dimension* of V is the number of elements in any basis of V.

For this definition to make sense, we need to check that any two bases for V have the same number of elements, but this turns out to be true (as one might expect).

The following important theorem is proven in linear algebra courses:

Theorem. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then*

$$\dim(\mathit{null}(T)) + \dim(\mathit{range}(T)) = n.$$

Homework 9. Due Friday, March 5, 2010.

H.9.1.a. Suppose that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Find a basis for null(T).

b. Find a basis for range(T).

c. What are the dimensions of null(T) and range(T)? Do these dimensions agree with the theorem stated above?