Mathematics 108A: Reduction and Extension Theorems

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Recall the basic lemma from Chapter 1 of the text:

Linear Dependence Lemma. *Suppose that* $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ *is a linearly dependent list of vectors in a finite-dimensional vector space* V *and that* $\mathbf{v}_1 \neq \mathbf{0}$*. Then there exists* $j \in \{2, \ldots, m\}$ *such that* $\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$*. Moreover,*

$$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

Assuming this lemma, we were able to prove the MAIN RESULT of Chapter 1:

Replacement Theorem. *If* V *is a finite-dimensional vector space over* F*,* $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ *is a linearly independent list of elements of* V *and* V *is the span of a list* $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$*, then* $m \leq n$*.*

You should know how to prove both the Linear Dependence Lemma (as worked out in the online key to Quiz C) and the Replacement Theorem (as worked out in the key to Quiz D). The Linear Dependence Lemma can be used to prove the most difficult of the other theorems on linear dependence and independence. For example, here are two proofs that you should learn how to reconstruct:

Problem 1. Use the Linear Dependence Lemma to prove the following:

Reduction Theorem. *Every spanning list in a vector space* V *can be reduced to a basis.*

Idea: The idea of the proof is to start with a spanning list and throw away elements until you have a basis. As long as you don't have a basis, the Linear Dependence Lemma says that you can throw something away.

Proof: Suppose that $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Start with the list $L = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

For each j, $1 \leq j \leq n$ ask whether $\mathbf{v}_j$ is in the span of the previous elements of the list. If so, throw it away, obtaining a new list. If not, keep $\mathbf{v}_j$ in the list.

Repeat this procedure n times obtaining a new list L which spans V. The Linear Dependence Lemma now implies that this list is linearly independent. (If not, one of the elements would have been in the span of the previous elements and would have been thrown away.) Hence L is a basis for V.

Problem 2. Use the Linear Dependence Lemma to prove the following:

Extension Theorem. *Every linearly independent list in a finite-dimensional vector space* V *can be extended to a basis.*

Idea: The idea of the proof is to suppose that $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. One by one, add the $\mathbf{w}_i$'s to the list L, throwing away any additions that make the list linearly dependent (using the Linear Dependence Lemma).

Proof: Suppose that $L = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$.

Step 1. If $\mathbf{w}_1$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_1$ to the end of the list. $\mathbf{w}_1$ is in the span of the resulting list and the Linear Dependence Lemma shows that the list is linearly independent.

Step k. Suppose that we have constructed a linearly independent list L extending $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ such that $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$ are in the span of L. If $\mathbf{w}_k$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_k$ to the list. $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are in the span of the new list and the list is linearly independent by the Linear Dependence Lemma.

After n steps, one obtains a list which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$, spans V and is linear independent. The resulting list is a basis which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$.