2. Complex Analytic Functions

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Recall that if A and B are sets, a function $f : A \to B$ is a rule which assigns to each element $a \in A$ a unique element $f(a) \in B$. In this course, we will usually be concerned with complex-valued functions of a complex variable, functions $f : U \to \mathbb{C}$, where U is an open subset of \mathbb{C} . For such a function, we will often write

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where u(x, y) and v(x, y) are the real and imaginary parts of f. We can think of the complex-valued function f as specified by these two real-valued functions u and v.

For example, if $f: \mathbb{C} \to \mathbb{C}$ is defined by $w = f(z) = z^2$, then

$$z^{2} = (x + iy)^{2} = x^{2} - y^{2} + 2ixy \quad \Rightarrow \quad \begin{cases} u(x, y) = x^{2} - y^{2}, \\ v(x, y) = 2xy. \end{cases}$$

We can think of this as defining a transformation

$$u = x^2 - y^2, \quad v = 2xy$$

from the (x, y)-plane to the (u, v)-plane.

Our goal is to study complex analytic functions $f: U \to \mathbb{C}$, functions which have a complex derivative at each point of U. We will see that the existence of a complex derivative at every point is far more restrictive than the existence of derivatives of real valued functions. We will also see that a function $f: U \to \mathbb{C}$ is complex analytic if and only if its component functions u and v have continuous partial derivatives and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Moreover, u(x, y) and v(x, y) automatically have arbitrarily many derivatives and satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These equations have many practical applications. For example, the real part of an analytic function can be used to model steady-state temperature distributions in regions of the plane.

1 Convergence and continuity

To properly deal with complex-valued functions, we need to understand limits and continuity. These are similar to the same concepts for real-valued functions which are studied informally in calculus or more carefully in real analysis courses such as Math 117 (see [5]). The simplest of the definitions is that of limit of a complex sequence.

Definition 1. A sequence (z_n) of complex numbers is said to *converge* to a complex number z if for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n \in \mathbb{N}$$
 and $n > N \Rightarrow |z_n - z| < \varepsilon$.

In this case, we write $z = \lim z_n$. A sequence (z_n) of real numbers which does not converge to a real number is said to *diverge*.

Example 1. We claim that the sequence (z_n) defined by $z_n = 1/n$ converges to 0. Indeed, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $N > 1/\varepsilon$ and thus $1/N < \varepsilon$ by the so-called Archimedean property of the real numbers. It follows that

$$n > N \quad \Rightarrow \quad 0 < \frac{1}{n} < \frac{1}{N} \quad \Rightarrow \quad |z_n - 0| = \left|\frac{1}{n} - 0\right| < \varepsilon.$$

Using the same technique, you could show that the sequence (z_n) defined by $z_n = c/n$ converges to 0, whenever c is a complex number.

Example 2. On the other hand, the sequence (z_n) defined by $z_n = i^n$ diverges. We can prove this by contradiction. Suppose that this sequence (z_n) were to converge to z. We could then take $\varepsilon = 1$, and there would exist $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |z_n - z| < 1.$$

But then if n > N and n is even, we would have $z_{n+2} = (i^2)z_n = -z_n$. Since $|z_n| = 1$,

$$2 = |z_n - z_{n+2}| \le |z_n - z| + |z - z_{n+2}| < 1 + 1 = 2$$

a contradiction.

Example 3. Suppose that $z_n = a^n$, where $a \in \mathbb{R}$ and 0 < a < 1. Then

$$\frac{1}{a} = 1 + b, \quad \text{where} \quad b > 0,$$

and by the binomial formula

$$\left(\frac{1}{a}\right)^n = (1+b)^n = 1 + nb + \dots + b^n \ge nb,$$

 \mathbf{SO}

$$a^n \le \frac{1}{bn} = \frac{c}{n}$$
, where $c = \frac{1}{b}$.

Now the argument for Example 1 can be applied with the result that

$$\lim z_n = \lim (a^n) = 0.$$

Finally if $c \in \mathbb{C}$ and |c| < 1, then

$$|c^n - 0| \le |c|^n \quad \Rightarrow \quad \lim c^n = 0.$$

Proposition 1. Suppose that (z_n) and (w_n) are convergent sequences of complex numbers with $\lim z_n = z$ and $\lim w_n = w$. Then

- 1. $(z_n + w_n)$ converges and $\lim(z_n + w_n) = z + w$,
- 2. $(z_n w_n)$ converges and $\lim(z_n w_n) = zw$,
- 3. (z_n/w_n) converges and $\lim(z_n/w_n) = z/w$, provided $w_n \neq 0$ for all n and $w \neq 0$.

This theorem is proven in Math 117 for sequences of real numbers, and exactly the same proof holds for sequences of complex numbers. For example, to prove part I, we let $\varepsilon > 0$ be given. Since (z_n) converges to z, there exists an $N_1 \in \mathbb{N}$ such that

$$n \in \mathbb{N}$$
 and $n > N_1 \Rightarrow |z_n - z| < \frac{\varepsilon}{2}$.

Since (w_n) converges to w, there exists an $N_2 \in \mathbb{N}$ such that

$$n \in \mathbb{N}$$
 and $n > N_2 \Rightarrow |w_n - w| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$. Then using the triangle inequality, we conclude that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |(z_n + w_n) - (z + w)| \le |z_n - z| + |w_n - w| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is exactly what we needed to prove. Similar arguments are used to prove the other parts of the Proposition.

Example 4. Suppose we want to investigate the convergence of the sequence (z_n) defined by

$$z_n = \frac{2n+3+i}{n+5-4i}.$$

We can rewrite this as

$$z_n = \frac{2 + (3+i)/n}{1 + (5-4i)/n}.$$

By Example 1, we see that as $n \to \infty$, (3+i)/n and (5-4i)/n converge to zero. We can then use the above Proposition to establish that

$$\lim(2+(3+i)/n) = 2$$
, $\lim(1+(5-4i)/n) = 1$ and $\lim z_n = 2$.

Similarly, we can use the Proposition to show that the sequence (z_n) defined by

$$z_n = \frac{2n^2 + (3+5i)n + (6-i)}{3n^2 + (5+i)n + (2+3i)}$$

converges to z = 2/3.

Example 5. Suppose that (z_n) is the sequence of complex numbers defined by

$$z_1 = 1.5, \quad z_{n+1} = f(z_n) = \frac{z_n}{2} + \frac{1}{z_n}, \quad \text{for } n \in \mathbb{N}.$$
 (1)

If this sequence has a limit, Proposition 1 tells us what the limit must be. Indeed, by induction one sees that $z_n \in \mathbb{R}$ and $z_n > 0$ for all $n \in \mathbb{N}$. Moreover, using calculus, one can show that $x \in \mathbb{R} \Rightarrow f(x) \ge \sqrt{2}$. Thus if $z = \lim z_n$, it follows from Proposition 1 that

$$z = \lim z_{n+1} = \frac{\lim z_n}{2} + \frac{1}{\lim z_n} = \frac{z}{2} + \frac{1}{z} \quad \Rightarrow \quad \frac{z}{2} = \frac{1}{z} \quad \Rightarrow \quad z^2 = 2.$$

Thus we see that $z = \sqrt{2}$. We remark that (1) provides a good numerical method for finding the square root of two.

We next turn to the notion of limits of functions. If z_0 is a complex number, a *deleted open ball* about z_0 is a set of the form

$$N(z_0; \varepsilon) - \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\},\$$

for some $\varepsilon > 0$.

Definition 2. Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood of z_0 lies within D. A complex number w_0 is the *limit* of a function $f: D \to \mathbb{C}$ at z_0 if

$$z_n \in D$$
, $z_n \neq z_0$ and $\lim z_n = z \Rightarrow \lim f(z_n) = w_0$.

In this case, we write

$$\lim_{z \to z_0} f(z) = w_0.$$

Some authors, including [6], prefer an alternate definition which turns out to be equivalent:

Definition 2'. Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood of z_0 lies within D. A complex number w_0 is the *limit* of a function $f: D \to \mathbb{C}$ at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon.$$

With either Definition 2 or 2', it is important that the function f need be defined only on a deleted open ball about z_0 . Indeed, we will often take limits at z_0 when f is not defined at z_0 . The following proposition follows immediately from Definition 2 and Proposition 1:

Proposition 2. Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood of z_0 lies within D. Suppose that $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ are functions such that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$. Then

- 1. $\lim_{z \to z_0} (f(z) + g(z)) = w_0 + w_1$,
- 2. $\lim_{z \to z_0} (f(z)g(z)) = w_0 w_1$,
- 3. if $g(z) \neq 0$ for $z \in D$ and $w_1 \neq 0$, then $\lim_{z \to z_0} (f(z)/g(z)) = w_0/w_1$.

Definition 3. Suppose that $D \subseteq \mathbb{C}$, that $f : D \to \mathbb{C}$, and z_0 is a point of D such that $N(z_0; \varepsilon) \subseteq D$ for some $\varepsilon > 0$. Then f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The following proposition follows immediately from this definition and Proposition 2:

Proposition 3. Suppose that $D \subseteq \mathbb{C}$ and that $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ are continuous at $z_0 \in D$. Then the functions f + g and $f \cdot g$ are also continuous at z_0 . Moreover, if $g(z) \neq 0$ for $z \in D$, then the quotient f/g is also continuous at z_0 .

It is quite easy to show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by f(z) = z is continuous at every $z_0 \in \mathbb{C}$. It then follows from Proposition 3 that every polynomial function

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

with complex coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$, is continuous at every $z_0 \in \mathbb{C}$. Suppose that

$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0 = 0$$

is a second polynomial with complex coefficients and that S be the finite set of points at which Q(z) vanishes. Let $D = \mathbb{C} - S$. Then it also follows from Proposition 3 that the *rational function* $R: D \to \mathbb{C}$ defined by

$$R(z) = \frac{P(z)}{Q(z)}$$

is continuous at every point of D. Thus we can construct many examples of continuous functions. Moreover, it is easy to calculate the limits of continuous functions, because if f is continuous at z_0 then

$$\lim_{z \to z_0} f(z) = f(z_0)$$

But there are also numerous cases in which limits do not exist. For an important example, suppose that $D = \mathbb{C} - \{0\}$ and that $f : D \to \mathbb{C}$ is defined by

$$f(x+iy) = f(z) = \frac{\bar{z}}{z} = \frac{x-iy}{x+iy} = \frac{(x-iy)^2}{x^2+y^2} = \frac{x^2-y^2-2xyi}{x^2+y^2}.$$

Then if $(z_n = x_n + iy_n)$ is a sequence in D which lies on the x-axis, $y_n = 0 \Rightarrow f(z_n) = 1$, while if $(z_n = x_n + iy_n)$ is a sequence in D which lies on the y-axis, $x_n = 0 \Rightarrow f(z_n) = -1$. Thus $\lim_{z\to 0} f(z)$ does not exist.

Here is another example. Recall that $\operatorname{Arg}(z)$ is the unique value of the multivalued angle function $\operatorname{arg}(z)$ which lies in the interval $(-\pi, \pi]$. This defines a function

$$\operatorname{Arg}: \mathbb{C} - \{0\} \to (-\pi, \pi]$$

which we use to define the logarithm

$$\operatorname{Log}: \mathbb{C} - \{0\} \to \mathbb{C} \quad \text{by} \quad \operatorname{Log}(z) = \operatorname{Log}|z| + i\operatorname{Arg}(z).$$
 (2)

As we saw in the notes on Complex numbers, if ex : $\mathbb{C} \to \mathbb{C}$ is the function defined by $ex(z) = e^z$, then

$$\exp \circ \operatorname{Log}(z) = z, \qquad \operatorname{Log} \circ \exp(w) = w,$$

when the composition is defined. If $(z_n = x_n + iy_n)$ is a sequence of complex numbers such that $y_n > 0$ and $\lim z_n = -1$, then $\lim \operatorname{Log}(z_n) = \pi$, while if $(z_n = x_n + iy_n)$ is a sequence such that $y_n < 0$ and $\lim z_n = -1$, then $\lim \operatorname{Log}(z_n) = -\pi$. Thus

$$\lim_{z \to -1} \operatorname{Log}(z)$$

does not exist, and the function Log defined by (2) fails to be continuous at z = -1.

On the other hand, one can show that the restricted function

$$\operatorname{Log}: \mathbb{C} - \{x \in \mathbb{R} : x \le 0\} \to \mathbb{C}, \text{ defined by } \operatorname{Log}(z) = \operatorname{Log}|z| + i\operatorname{Arg}(z),$$

is in fact continuous, because we have excised the points of discontinuity from the domain.

2 Complex derivatives and analyticity

The notion of derivative for a complex-valued function looks superficially similar to the similar definition for real-valued functions, but we will see that it has far stronger implications:

Definition 1. Suppose that $f: U \to \mathbb{C}$ is a complex valued function, where U is an open subset of \mathbb{C} and $z_0 \in U$. Then the *complex derivative* of f at z_0 is

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$
(3)

if this limit exists. We say that f is *differentiable* at z_0 if it has a complex derivative at z_0 .

Definition 2. A function $f: U \to \mathbb{C}$, where U is an open subset of \mathbb{C} , is said to be *complex analytic* or *holomorphic* if it has a derivative at every $z_0 \in U$. If $U = \mathbb{C}$, we say that the function is *entire*.

Example 1. If $f : \mathbb{C} \to \mathbb{C}$ is the function defined by $f(z) = z^n$, then it follows from the binomial formula that

$$f(z_0 + \Delta z) = z_0^n + n z_0^{n-1} \Delta z + \binom{n}{2} z_0^{n-2} (\Delta z)^2 + \dots + (\Delta z)^n,$$

 \mathbf{SO}

$$f(z_0 + \Delta z) - f(z_0) = n z_0^{n-1} \Delta z + \binom{n}{2} z_0^{n-2} (\Delta z)^2 + \dots + (\Delta z)^n$$

and

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = nz_0^{n-1} + \binom{n}{2} z_0^{n-2} (\Delta z) + \dots + (\Delta z)^{n-1}$$

Taking the limit as $\Delta z \to 0$ yields

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = n z_0^{n-1}.$$

Thus in this case, the derivative of f exists at every $z_0\in\mathbb{C}$ and is given by the familiar formula

$$f'(z_0) = n z_0^{n-1}$$
.

Thus f is complex analytic on the entire complex plane, that is, it is an entire function.

Example 2. If $f : \mathbb{C} \to \mathbb{C}$ is the function defined by $f(z) = \overline{z}$, then

$$f(z_0 + \Delta z) - f(z_0) = \overline{(z_0 + \Delta z)} - \overline{z}_0 = \overline{\Delta z}.$$

Thus

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{\Delta z}}{\overline{\Delta z}}$$

But

$$\Delta z \in \mathbb{R} \quad \Rightarrow \quad \frac{\Delta z}{\Delta z} = 1 \quad \text{while} \quad \Delta z \in i\mathbb{R} \quad \Rightarrow \quad \frac{\Delta z}{\Delta z} = -1$$

 \mathbf{SO}

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

cannot exist in this case, so f is not differentiable at any $z_0 \in \mathbb{C}$, even though if we divide f into real and imaginary parts,

$$f(z) = f(x+iy) = u(x,y) + iv(x,y), \quad \text{then} \quad \begin{cases} u(x,y) = x, \\ v(x,y) = -y \end{cases}$$

and the component functions u and v have continuous partial derivatives of arbitrarily high order.

One can use Proposition 2 from §1 as the foundation for proving:

Proposition 1. If $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ are differentiable at $z_0 \in D$, then

- 1. f + g is differentiable at z_0 , and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$,
- 2. cf is differentiable at z_0 for any constant c, and $(cf)'(z_0) = cf'(z_0)$,
- 3. fg is differentiable at z_0 , and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,
- 4. if $g(z) \neq 0$ for $z \in D$, then f/g is differentiable at z_0 . and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$
(4)

Using Example 1 and this Proposition, it becomes straightforward to show that any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

with complex coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ is differentiable at every point $z_0 \in \mathbb{C}$, with complex derivative given by the formula

$$P'(z_0) = na_n z_0^{n-1} + n(n-1)a_{n-1} z_0^{n-2} + \dots + a_1.$$

Moreover, if

$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0 = 0$$

is a second polynomial with complex coefficients and S is the finite set of points at which Q(z) vanishes. It also follows from Proposition 1 that the rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

is differentiable at every point of $D = \mathbb{C} - S$. In this case, we would say that R is complex analytic except at the points of S, and we can use (4) to find the derivative at any point of $\mathbb{C} - S$.

We can also prove a version of the chain rule for complex derivatives:

Proposition 2. Suppose that U and V are open subsets of the complex plane \mathbb{C} and that $f: U \to V$ and $g: V \to \mathbb{C}$ are differentiable at $z_0 \in U$ and $f(z_0) \in V$ respectively. Then the composition $g \circ f: U \to \mathbb{C}$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Here is a simple application of the chain rule: We will see in the next section that the function

$$g: \mathbb{C} \to \mathbb{C}$$
 defined by $g(x+iy) = e^z = e^x(\cos y + i \sin y)$

is differentiable at every point of \mathbb{C} and $g'(z) = e^z$. If $f : \mathbb{C} \to \mathbb{C}$ is defined by f(w) = aw, where $a \in \mathbb{C}$, then $g \circ f(z) = e^{az}$, and it follows from the chain rule that

$$(g \circ f)'(z) = g'(f(z))f'(z) = e^{az}a = ae^{az}$$

Finally, just as in the real case, it turns out that a function which has a complex derivative at a point automatically is continuous at that point:

Proposition 3. If $f: D \to \mathbb{C}$ is differentiable at $z_0 \in D$, then f is continuous at z_0 as well.

To prove this, we let $z = z_0 + \Delta z$ in (3) so that $\Delta z = z - z_0$. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Thus

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0)$$
$$= f'(z_0) \lim_{z \to z_0} (z - z_0) = 0.$$

But this immediately implies that f is continuous at z_0 .

Thus if D is an open subset of \mathbb{C} and $f: D \to \mathbb{C}$ is complex analytic, then f is continuous. It is a little harder (theorem of Goursat) to show that a complex analytic function $f: D \to \mathbb{C}$ automatically has a continuous derivative $f': D \to \mathbb{C}$. Later we will see via the Cauchy integral theorem that in fact a complex analytic function has continuous derivatives of arbitrarily high order. Thus existence of complex derivatives is far stronger than the existence of ordinary derivatives of real valued functions.

3 The Cauchy-Riemann equations

Suppose that the function $f: D \to \mathbb{C}$ has a complex derivative at $z_0 = x_0 + iy_0 \in D$. In the definition of complex derivative, we can let Δz approach zero along the x-axis, that is, we can set $\Delta z = h \in \mathbb{R}$. In that case, (3) becomes

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

where h ranges only over \mathbb{R} . We recognize that this is just the partial derivative of the vector valued function f(x + iy) = u(x, y) + iv(x, y) with respect to x:

$$f'(z_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$
 (5)

On the other hand, we can also let Δz approach zero along the *y*-axis, that is, we can set $\Delta z = ik$, where $k \in \mathbb{R}$. In this case, we find that

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0 + ik) - f(z_0)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0) = \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$
 (6)

Since the two expressions (5) and (6) must be equal, we must have

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

These are known as the *Cauchy-Riemann equations* in honor of Augustin Cauchy (1789-1859) and Bernard Riemann (1826-1866), although these equations had actually appeared earlier in work of d'Alembert and Euler on fluid motion (as explained in Chapter 26 of [4]).

Conversely, we have the following key theorem:

Cauchy-Riemann Theorem. Suppose that U is an open subset of \mathbb{C} and the complex-valued function $f: U \to \mathbb{C}$ can be expressed in terms of real and imaginary parts as

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where u(x, y) and v(x, y) have continuous first order partial derivatives on U which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
(7)

Then f is a complex analytic function on U and its derivative is given by the formula

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$
(8)

We sketch a proof of this following Ahlfors [1], page 26. It is proven in calculus courses (or more rigorously in real analysis courses) that when u(x, y) and v(x, y) have continuous partial derivatives,

$$u(x+h,y+k) - u(x,y) = \frac{\partial u}{\partial x}(x,y)h + \frac{\partial u}{\partial y}(x,y)k + \varepsilon_1$$

and

$$v(x+h,y+k) - v(x,y) = \frac{\partial v}{\partial x}(x,y)h + \frac{\partial v}{\partial y}(x,y)k + \varepsilon_2,$$

where

$$\frac{\varepsilon_1}{\sqrt{h^2+k^2}} \to 0 \quad \text{and} \quad \frac{\varepsilon_2}{\sqrt{h^2+k^2}} \to 0 \quad \text{as} \quad \sqrt{h^2+k^2} \to 0.$$

Using (7), we can rewrite the above equations as

$$\begin{pmatrix} u(x+h,y+k)\\v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y)\\v(x,y) \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y\\\partial v\partial x & \partial v/\partial y \end{pmatrix} (x,y) \begin{pmatrix} h\\k \end{pmatrix} + \begin{pmatrix} \varepsilon_1\\\varepsilon_2 \end{pmatrix}.$$
 (9)

The key point now is that the Jacobian matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v\partial x & \partial v/\partial y \end{pmatrix} (x,y) \quad \text{represents a complex matrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

if and only if the Cauchy-Riemann equations are satisfied. Thus if the Cauchy-Riemann equations are satisfied, we can rewrite (9) as

$$\begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} (x,y) \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

One can check that the two components of this last equation are the real and imaginary parts of the complex equation

$$f(z + (h + ik)) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h + ik) + \varepsilon_1 + i\varepsilon_2.$$

But this implies that

$$\lim_{h+ik\to 0} \left(\frac{f(z+(h+ik))-f(z)}{h+ik} \right) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}(x,y),$$

so f is analytic and its derivative is given by (8).

Application 1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is the exponential function defined by

$$f(x + iy) = e^{z} = e^{x}(\cos y + i\sin y) = u(x, y) + iv(x, y),$$

where $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Then the real and imaginary parts of f are continuously differentiable, and

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$
 and $\frac{\partial u}{\partial y} = -e^x \sin y = \frac{\partial v}{\partial x}$,

so the theorem implies that f is a complex analytic function. Moreover,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = e^x \cos y + ie^x \sin y, \quad \text{so} \quad \frac{d}{dz}(e^z) = e^z.$$

Exercise B. a. Use the chain rule to express the partial derivatives of u and v with respect to x and y in terms of the partial derivatives with respect to the polar coordinates (r, θ) , where

$$x = r \cos \theta, \qquad y = r \sin \theta.$$

b. Use the expressions you obtained to rewrite the Cauchy-Riemann equations in terms of polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$
(10)

Application 2. Suppose that

 $f = \text{Log}: \mathbb{C} - \{x \in \mathbb{R} : x \le 0\} \to \mathbb{C}, \text{ is defined by } \text{Log}(z) = \text{Log}|z| + i\text{Arg}(z),$ In this case,

$$u(x,y) = \operatorname{Log}\left(\sqrt{x^2 + y^2}\right), \quad v(x,y) = \operatorname{arg}(x + iy).$$

In terms of polar coordinates,

$$u(r,\theta) = Log(r), \quad v(r,\theta) = \theta,$$

 \mathbf{so}

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = 0, \quad \frac{\partial u}{\partial \theta} = 1.$$

It therefore follows from the polar coordinate form of the Cauchy-Riemann equations (10), together with the Theorem, that f = Log is a complex analytic function of z, and using (8) we find its derivative:

$$\begin{aligned} f'(z) &= \frac{\partial}{\partial x} \mathrm{Log}\left(\sqrt{x^2 + y^2}\right) - i\frac{\partial}{\partial y} \mathrm{Log}\left(\sqrt{x^2 + y^2}\right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \left(\sqrt{x^2 + y^2}\right) - \frac{i}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \left(\sqrt{x^2 + y^2}\right) \\ &= \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{\bar{z}z} = \frac{1}{z}. \end{aligned}$$

The following Proposition follows from Proposition 2 in §5 of the notes on complex numbers:

Proposition 1. Suppose that D is a connected open subset of \mathbb{C} and that $f: D \to \mathbb{C}$ is a complex analytic function such that $f': D \to \mathbb{C}$ is continuous. Then

$$f'(z) = 0$$
 for all $z \in D \Rightarrow f(z) \equiv c$,

where c is a constant.

Indeed, if f'(z) = 0 for all $z \in D$, then it follows from (8) and the Cauchy-Riemann equations that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Since D is connected, it follows from Proposition 2 in the Notes on Complex Numbers (or from Theorem 1, page 40 in [6]) that u and v are both constant.

Many properties of complex analytic functions can be derived from the Cauchy-Riemann equations. For example:

Proposition 2. Suppose that D is a connected open subset of |mathbbC| and that $f: U \to \mathbb{C}$ is a complex analytic function such that $f': D \to \mathbb{C}$ is continuous. If the real part of f is constant, then f itself is constant.

Proof: Suppose that f(x + iy) = u(x, y) + iv(x, y), so that u is the real part of f. Then

$$u \text{ constant} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

It then follows from the Cauchy-Riemann equations that

$$\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

It therefore follows from Proposition 2 in the Notes on Complex Numbers that v is constant. Thus f itself is constant. QED

4 Fluid motion in the plane

In his research on complex analysis, Riemann utilized physical models to buttress his intuition, as emphasized by Felix Klein in his classic treatise [3] on Riemann's theory of complex functions. One of the models Riemann used was that of a fluid flow tangent to an electric field in the (x, y)-plane, a flow which turns out to be both incompressible and irrotational.

One can represent the velocity of a fluid in an open subset U of the $(\boldsymbol{x},\boldsymbol{y})\text{-}$ plane by a vector field

$$\mathbf{V}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} : U \longrightarrow \mathbb{R},$$

where **i** and **j** are the perpendicular unit-length vectors pointing in the x and y coordinate directions. We say that **V** has continuous first-order partial derivatives if its component functions M and N have continuous first-order partial derivatives. Such a vector field \mathbb{V} (or the corresponding fluid) is said to be *incompressible* if

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0, \tag{11}$$

and *irrotational* if

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0. \tag{12}$$

An important fact is that we can rotate the vector field ${\bf V}$ counterclockwise through 90 degrees, obtaining

$$\star \mathbf{V} = -N(x, y)\mathbf{i} + M(x, y)\mathbf{j},\tag{13}$$

and one can check that

 $V \text{ is irrotational} \quad \Leftrightarrow \quad \star V \text{ is incompressible},$ $V \text{ is incompressible} \quad \Leftrightarrow \quad \star V \text{ is irrotational}.$

We will see later that the velocity field for a steady-state fluid of constant density is incompressible so long as no fluid is being created or destroyed. On the other hand, if U is either \mathbb{C} or an open ball in \mathbb{C} then a vector field is irrotational if and only if it is the gradient of a function:

Poincaré Lemma. Suppose that $U = \mathbb{C}$ or

$$U = N((x_0, y_0); R) = \{x + iy \in \mathbb{C} : d((x, y), (x_0, y_0)) < R\},\$$

for some $(x_0, y_0) \in \mathbb{C}$ and some R > 0, and that **V** is is a vector field on U with continuous first-order partial derivatives. Then

$$\mathbf{V} = \nabla u, \quad \text{for some } u : U \to \mathbb{R} \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Here the condition $\mathbf{V} = \nabla u$ means that

$$\frac{\partial u}{\partial x}(x,y) = M(x,y), \quad \frac{\partial u}{\partial y}(x,y) = N(x,y).$$
 (14)

One direction of the proof is easy. If $\mathbf{V} = \nabla u$, then (14) and equality of mixed partials yields

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = 0.$$

We will defer the proof of the other direction of the Poincaré Lemma until we have discussed contour integrals. For now we simply note that one constructs the function $u: U \to \mathbb{R}$ such that $\mathbf{V} = \nabla u$ by the *method of exact differentials*. To express this in the language of differentials, we let

$$d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j}, \text{ so that } \nabla u \cdot d\mathbf{x} = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$

and $\mathbf{V}(x, y) \cdot d\mathbf{x} = (M\mathbf{i} + N\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = Mdx + Ndy,$

and hence

 $\mathbf{V} = \nabla u \quad \Leftrightarrow \quad Mdx + Ndy = du.$

Example 1. Suppose that

$$\mathbf{V}(x,y) = (4x^3 - 12xy^2)\mathbf{i} + (-12x^2y + 4y^3)\mathbf{j}$$

so that $Mdx + Ndy = (4x^3 - 12xy^2)dx + (-12x^2y + 4y^3)dy.$

Then

$$\frac{\partial N}{\partial x} = -12x^2 + 12y^2 = \frac{\partial M}{\partial y},$$

so $\mathbf{V} = \nabla u$, for some function u. But then

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \quad \Rightarrow \quad u(x,y) = x^4 - 6x^2y^2 + g(y),$$

where g(y) is a function of y alone, while

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \quad \Rightarrow \quad u(x,y) = x^4 - 6x^2y^2 + y^4 + c$$

where c is a constant. Thus **V** determines u up to a constant. We call u a *potential* for the fluid flow **V**.

Suppose now that $\mathbf{V}: U \to \mathbb{R}$ is both incompressible and irrotational. If U is the entire plane or an open ball, it follows from the Poincaré Lemma that \mathbf{V} has a potential u and substituting (14) into (11) yields the Laplace equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
(15)

Definition. A function $u : U \to \mathbb{R}$ with continuous first and second order partial derivatives is said to be *harmonic* if u satisfies (15).

Thus harmonic functions are exactly the potentials for irrotational incompressible fluid flows in the plane. What is important for complex analysis is that if

$$f(x+iy) = u(x,y) + iv(x,y)$$

is a complex analytic function defined on an open subset U of the complex plane \mathbb{C} , then u and v are harmonic functions. Not only that, but a harmonic function $u: U \to \mathbb{C}$ is the real part of a complex analytic function, at least if Uis the entire complex plane or an open ball in the complex plane. Indeed, if uis harmonic, then its gradient

$$\mathbf{V} = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}$$

is both irrotational and incompressible. But then

$$\star \mathbf{V} = -\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j}$$

is also both irrotational and incompressible. If U is the entire complex plane or an open ball in the complex plane, we can apply the Poincarë Lemma and construct a potential $v: U \to \mathbb{R}$ for $\star \mathbf{V}$, so that

$$-\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j} = \star \mathbf{V} = \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}.$$

But then u and v satisfy the Cauchy-Riemann equations, and we have proven:

Theorem. Suppose that $U = \mathbb{C}$ or an open ball within U and that $u : U \to \mathbb{R}$ is a harmonic function on U. Then up to addition of a constant, there is a unique harmonic function v(x, y) such that

$$f(x+iy) = u(x,y) + iv(x,y), \text{ for } (x,y) \in U.$$

We call v the harmonic conjugate of u.

Example 2. Suppose that

$$u(x,y) = x^4 - 6x^2y^2 + y^4$$

a function which is easily verified to be harmonic. Then

$$\mathbf{V} = \nabla u = (4x^3 - 12xy^2)\mathbf{i} + (-12x^2y + 4y^3)\mathbf{j},$$

so that

$$\star \mathbf{V} = (12x^2y - 4y^3)\mathbf{i} + (4x^3 - 12xy^2)\mathbf{j},$$

and we can find the harmonic conjugate v as follows:

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 \quad \Rightarrow \quad v(x,y) = 4x^3y - 4xy^3 + g(y),$$

where g(y) is a function of y alone. Then

$$\frac{\partial v}{\partial y} = 4x^3 - 12xy^2 \quad \Rightarrow \quad u(x,y) = 4x^3y - 4xy^3 + c,$$

where c is a constant. Thus if we set c = 0,

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

= $x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3) = (x + iy)^4 = z^4$.

We have seen that the study of irrotational incompressible fluid motion in the plane inexorably leads to the Cauchy-Riemann equations of complex analysis. Following Klein [3], we can imagine an electric field $\mathbf{V}(x, y)$ in the (x, y)-plane produced by a finite number of charges located at the points $\{z_1, \ldots, z_k\}$ (which will be singular points for \mathbf{V}). Maxwell's equations from electricity and magnetism imply that \mathbf{V} is an incompressible irrotational flow on the open set

$$U = \mathbb{C} - \{z_1, \dots, z_k\}.$$

However, even if we assume that **V** has a potential u, U is not an open ball within \mathbb{C} , so we cannot apply the Poincaré Lemma to construct a harmonic conjugate v. This raises the question: Can we extend the Poincaré Lemma to more general connected open sets U?

Exercise C. Using Exercise B, show that we can write Laplace's equation in polar coordinates as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Example 3. It follows from Exercise C, that the only solutions to Laplace's equation which are radially symmetric are the solutions to the ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0.$$

If we set w = du/dr, this yields

$$\frac{dw}{dr} + \frac{w}{r} = 0 \quad \text{which has the solutions} \quad w = \frac{a}{r},$$

where a is a constant. This in turn implies that

$$u = a \operatorname{Log} r + b = a \operatorname{Log}|z| + b, \tag{16}$$

where a and b are constants; if b = 0 and a > 0, this is interpreted as a source at the origin z = 0 of strength a. In the electrostatic model, u is the potential produced by an electric charge placed at the origin.

But now we can ask the question: Does u have a harmonic conjugate? If we let

$$U = \mathbb{C} - \{ x \in \mathbb{R} : x \le 0 \},\$$

then the harmonic conjugate to u must be the imaginary part of the function Log we described before, which is given by

$$v(x,y) = \operatorname{Arg}(x+iy),$$

while if $U = \mathbb{C} - \{0\}$, there is no continuous harmonic conjugate. (The harmonic conjugate would have to be the "multivalued function" arg, but that of course is not a genuine function.) We will return to the question of when the harmonic conjugate to a given harmonic function exists when we study contour integrals.

Remark: The stereographic projection $\Phi : \mathbb{C} \to S^2 - \{N\} \subseteq \mathbb{R}^3$ allows us to extend this model of fluid flow to the surface of the sphere S^2 . Indeed, if $\phi : S^2 - \{N\} \to \mathbb{C}$ is the inverse to stereographic projection,

$$u \circ \phi : S^2 - \{N\} \longrightarrow \mathbb{C}$$

can be thought of as the potential for a fluid on $S^2 - \{N\}$, and the best behaved potentials are those which extend to the north pole N.

For example, we could take the harmonic function $u = a \operatorname{Log}|z|$ of Example 3. How does this fluid flow behave near the north pole N? To answer that question, we write u in terms of the coordinate w = 1/z which is well behaved near N:

$$u = a \operatorname{Log}|z| = a \operatorname{Log}|1/w| = -a \operatorname{Log}|w|.$$

Thus if a > 0 a source of strength a at the origin z = 0 is balanced by a sink of strength a at $z = \infty$.

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